1.2 The action and the Euler Lagrange equations

- The action

$$S[\boldsymbol{r}(t)] = \int_{t_1}^{t_2} dt L(\boldsymbol{r}, \dot{\boldsymbol{r}}, t) \tag{1.31}$$

takes an arbitrary path $\boldsymbol{r}(t)$ (which may not satisfy the EOM) and returns a number. It is called a *functional*.

- The action principle says that the path $\boldsymbol{r}(t)$ that satisfies the EOM (sometimes called the the classical or "on-shell" path) is an extremum the action[1]. This means that if we replace the on-shell path $\boldsymbol{r}(t)$ with

$$\boldsymbol{r}(t) \to \boldsymbol{r}(t) + \delta\boldsymbol{r}(t) \tag{1.32}$$

for an arbitrary (small) function $\delta\boldsymbol{r}(t)$ that vanishes near t_1 and t_2 then the action is unchanged

$$S[\boldsymbol{r}(t) + \delta\boldsymbol{r}(t)] = S[\boldsymbol{r}(t)] \qquad \text{when } \boldsymbol{r}(t) \text{ is "on-shell", i.e. satisfies the EOM} \tag{1.33}$$

- Generally we define

$$\delta S[\boldsymbol{r}(t), \delta\boldsymbol{r}(t)] \equiv S[\boldsymbol{r}(t) + \delta\boldsymbol{r}(t)] - S[\boldsymbol{r}(t)] \tag{1.34}$$

and note that $\delta S[\boldsymbol{r}, \delta\boldsymbol{r}]$ depends on both the path and the variation. The requirement that $\delta S = 0$ determines the equation of motion. You should be able to prove that when $\delta S = 0$ for an arbitrary variation, the equations of motion are (in 1d)

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{\partial L}{\partial x} \tag{1.35}$$

- For a general set of coordinates $q^A = 1 \ldots N$ the equations of motion take the same form:

$$\delta S \equiv S[q(t) + \delta q(t)] - S[q(t)] = 0 \tag{1.36}$$

to first order in an arbitrary $\delta q(t)$. This leads to N equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = \frac{\partial L}{\partial q^A} \qquad A = 1 \ldots N \tag{1.37}$$

we call

$$p_A = \frac{\partial L}{\partial \dot{q}^A} \equiv \text{the canonical momentum conjugate to } q^A \tag{1.38}$$

$$F_A = \frac{\partial L}{\partial q^A} \equiv \text{the generalized force associated with } q^A \tag{1.39}$$

- If a coordinate q^A does not appear in the Lagrangian (but of course $\dot{q}^A$ does or it wouldn't appear at all), the variable is called *cyclic*. For a cyclic coordinate we have from the Euler Lagrange equations (Eq. (1.37))

$$\frac{dp_A}{dt} = 0 \tag{1.40}$$

i.e. p_A is a constant of the motion.

The hamiltonian function

- The hamiltonian (or energy) function (sometimes called the "first integral") is

$$h(q, \dot{q}, t) = p\dot{q} - L(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}}\dot{q} - L(q, \dot{q}, t) \tag{1.41}$$

and obey the equation of motion

$$\frac{dh}{dt} = -\frac{\partial L}{\partial t}\,. \tag{1.42}$$

$h(q, \dot{q}, t)$ is therefore constant if L does not depend explicitly on time.

[1] Sometimes for clarity we will put a bar, e.g., $\underline{\boldsymbol{r}}(t)$ to indicate that this path is on-shell, i.e. that it satisfies the EOM

- If more then one coordinate is involved then

$$h(q^A, \dot{q}^A, t) = \sum_A p_A \dot{q}^A - L \tag{1.43}$$

$$= \frac{\partial L}{\partial \dot{q}^A} \dot{q}^A - L \tag{1.44}$$

where we have and will *from now on* follow the summation convention, where repeated indices are summed over.

- We will distinguish the hamiltonian function $h(q, \dot{q}, t)$, which is a function of q, $\dot{q}$, and t, from the Hamiltonian $H(p, q, t)$ which is a function of q and p and t through the Legendre transform (more later). Thus $p_A(q, \dot{q}, t)$ in the hamiltonian function (Eq. (1.43)) is a function of the q and the $\dot{q}$, while in the Hamiltonian the $\dot{q}$ is a function of q and p.

- For a rather general Lagrangian

$$L = \frac{1}{2} a_{ij}(q)\, \dot{q}^i \dot{q}^j + b_i(q) \dot{q}^i - U(q)\,, \tag{1.45}$$

(which is the form of the Lagrangian for a particle in a magnetic field or gravity) the hamiltonian function is

$$h(\dot{q}, q, t) = \frac{1}{2} a_{ij}(q)\, \dot{q}^i \dot{q}^j + U(q) \tag{1.46}$$

The fact that the hamiltonian function is independent of b_i is closely related to the fact that magnetic fields do no work.

The period of one dimensional motion

- For one dimensional Lagrangian's of the form

$$L = \frac{1}{2} m(q)\, \dot{q}^2 - V_{\text{eff}}(q) \tag{1.47}$$

The first integral is

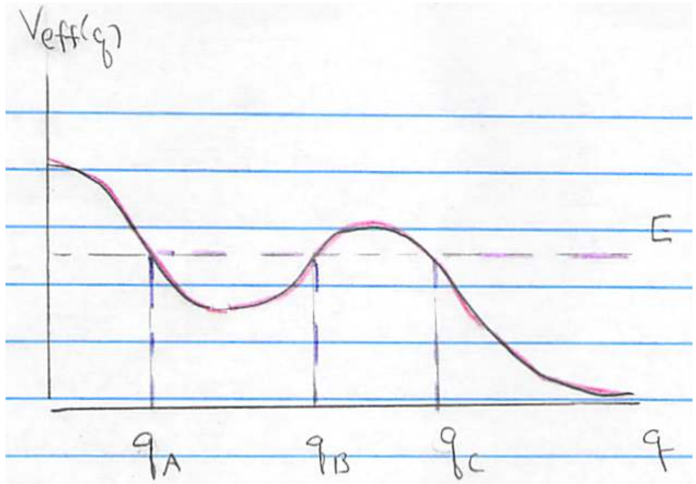
$$E = \frac{1}{2} m(q)\, \dot{q}^2 + V_{\text{eff}}(q) \tag{1.48}$$

You should be able to show that the this first integral equation can be used to determine $q(t)$ implicitly. Integrating from (t_0, q_0) to $(t, q(t))$ yields

$$\pm \int_{q_0}^{q(t)} dq \left(\frac{m(q)}{2(E - V_{\text{eff}}(q))} \right)^{1/2} = t - t_0\,, \tag{1.49}$$

which, when inverted, gives $q(t)$. The plus sign is when q is increasing in time, while the minus sign is when $q(t)$ is decreasing in time

- In a typical case the potential $V_{\text{eff}}(q)$ and energy E is shown below



For the specified energy, the motion is unbounded for $q > q_c$, and oscillates between when $q_A < q < q_B$. q_A, q_B and q_C are called turning points. The period $\mathcal{T}(E)$ is the time it takes to go from q_A to q_B and back. Thus half a period $\mathcal{T}(E)/2$ is the time it takes to go from q_A to q_B or

$$\frac{\mathcal{T}(E)}{2} = \int_{q_A}^{q_B} dq \left(\frac{m(q)}{2(E - V_{\text{eff}}(q))} \right)^{1/2} . \tag{1.50}$$