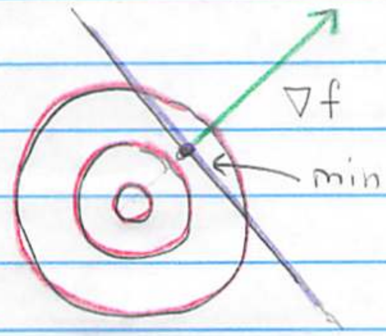
Constraints and Lagrange Multipliers

- let us pause to consider extremizing a function $U(x,y)$ with a constraint, i.e. a relation between x, y, $f(x,y)=0$

e.g.

$$U(x,y) = \frac{1}{2}(x^2+y^2) \qquad \underbrace{y + x - 1}_{f(x,y)} = 0$$



direction of motion (constant $f(x,y)$)

- This a very physical problem, e.g. a bead wants to fall down hill but is constrained to ride on the wire

- ∇f is perpendicular to lines of constant f (the direction of motion)

- It is clear that U reaches an extremum when there is no force along the beads motion (it is $\perp$ to lines of constant f). Thus we want

$$\underbrace{\nabla U}_{\text{force}} \propto \underbrace{\nabla f}_{\text{perpendicular to lines of constant } f.}$$

- Mathematically

(1) $$dU = \frac{\partial U}{\partial x}\,dx + \frac{\partial U}{\partial y}\,dy = 0$$

But dx and dy are not independent so $\partial u/\partial x$ and $\partial u/\partial y$ are not separately zero. But,

(2) $$df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy = 0 \quad \leftarrow \; f = \text{constant}$$

- So we can solve (1) and (2) by choosing a constant λ such that

$$\frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y} = 0$$

Then multiplying (2) by λ and adding (1)

$$dU - \lambda\, df = \left(\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y} + \lambda \frac{\partial f}{\partial y}\right)dy$$

- Now dx can be taken as the independent variable so

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial f}{\partial x} = 0$$

- Summarizing we are to solve:

$$\boxed{\begin{array}{ll} (1) & \dfrac{\partial u}{\partial x} + \lambda \dfrac{\partial f}{\partial x} = 0 \\ (2) & \dfrac{\partial u}{\partial y} + \lambda \dfrac{\partial f}{\partial y} = 0 \\ (3) & f(x,y) = 0 \end{array}}$$

determines x, y, λ

for x, y, λ to find the position x, y of the minimum (or extremum)

e.g for the example

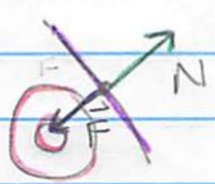
$$(1a) \quad x + \lambda = 0$$

$$(2a) \quad y + \lambda = 0 \qquad \Rightarrow \qquad x = y = 1/2, \quad \lambda = -1/2$$

$$(3a) \quad x + y - 1 = 0$$

- The normal force is (force of wire on bead) is determined by $-\lambda \nabla f$.

$$\vec{N} = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = -\lambda\left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$



$$= \left(\frac{1}{2}, \frac{1}{2}\right)$$

$\vec{F} = -\nabla u$

★ The upshot is instead of minimizing

$$U(x,y)$$

- We minimize with x, y, λ unconstrained $\hat{U}$

$$\boxed{\hat{U}(x,y,\lambda) \equiv U(x,y) + \lambda f(x,y)}$$

$$d\hat{U} = \left(\frac{\partial U}{\partial x} + \lambda \frac{\partial f}{\partial x}\right)dx + \left(\frac{\partial U}{\partial y} + \lambda \frac{\partial f}{\partial y}\right)dy + f(x,y)\,d\lambda$$

$$= 0$$

So find Eqs (1), (2), (3) of previous page as before.

★ This generalizes of N variables x^A and m-constraints $f^\alpha(x^A) = 0$. The potential $U(x^A)$ is minimized (with the constraint) by minimizing $\hat{U}$ with λ_α $\alpha = 1 \ldots m$ multipliers

$$\hat{U}(x^A, \lambda) \equiv U(x^A) + \sum_\alpha \lambda_\alpha f^\alpha(x^A)$$ (summed over α)

Leading to

$$\frac{\partial U}{\partial x^A} + \sum_\alpha \lambda_\alpha \frac{\partial f^\alpha}{\partial x^A} = 0 \quad \text{and} \quad f^\alpha(x^A) = 0$$