General Coordinates - The advantage of Lagrangians

- Consider a general coordinates transformation

$$q_A = f_A(\vec{x}_a, t) \qquad a = 1, \ldots, N$$

number of particles

invertible $\qquad A = 1, \ldots, 3N$

Total # independent variable

e.g

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

- Choose a new lagrangian, $\underline{L}$, so that

$$L(\vec{x}, \dot{\vec{x}}, t) = \underline{L}(q, \dot{q}, t)$$

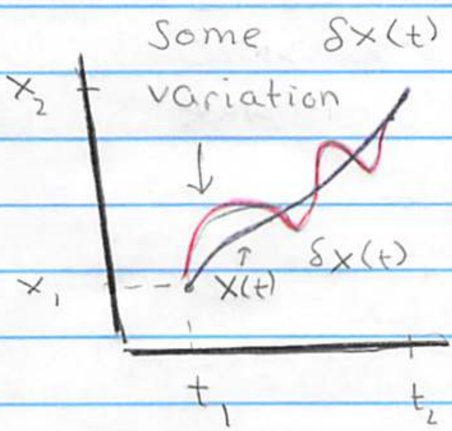
different functional form (so its techically a different func) but returns same value at corresponding points. We usually leave off the bar, but this can lead to confusion.

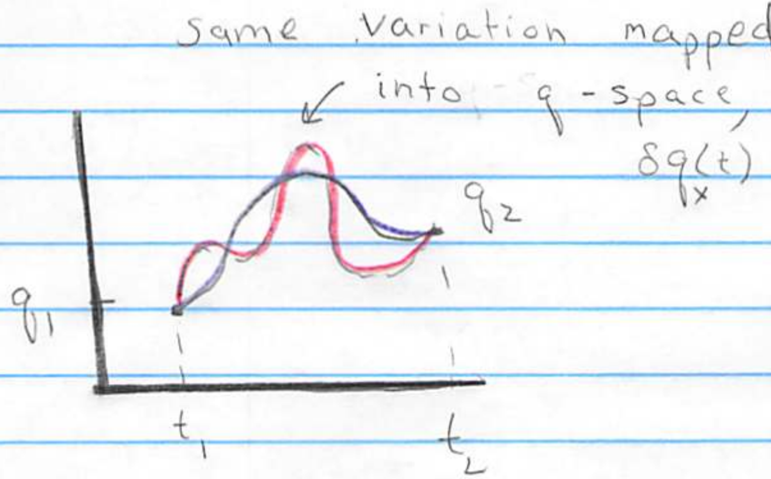
e.g

$$\underbrace{\frac{1}{2} m \dot{\vec{x}}^2}_{L(x, \dot{x}, t)} = \underbrace{\frac{1}{2} m \left(\dot{r}^2 + (r\dot{\theta})^2 \right)}_{\underline{L}(q, \dot{q}, t)}$$

• So

$$S[\vec{x}] = \int dt\, L(x, \dot{x}, t) = \int dt\, \underline{L}(q, \dot{q}, t) = \underline{S}[q]$$





So

$$S[\vec{x} + \delta\vec{x}] = \underline{S}[q + \delta q] = S[x] = \underline{S}[q]$$

i.e.

$$\boxed{S[q + \delta q] = \underline{S}[q]}$$

So the eom are the same and derived the same way. Briefly:

$$\underline{S}[q + \delta q] = \int dt\, \underline{L}\left(q + \delta q,\ \dot{q} + \frac{d\,\delta q}{dt},\ t\right)$$

And $\delta \underline{S} = \underline{S}[q + \delta q] - \underline{S}[q]$

$$\delta \underline{S} = \int dt\, \frac{\partial \underline{L}}{\partial q} + \frac{\partial \underline{L}}{\partial \dot{q}} \frac{d\,\delta q}{dt} \underset{\text{by parts}}{\Rightarrow} \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

Leading to

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

Or in general for coordinates q_A $A = 1, \ldots 3N$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = \frac{\partial L}{\partial q^A}}$$

In general call

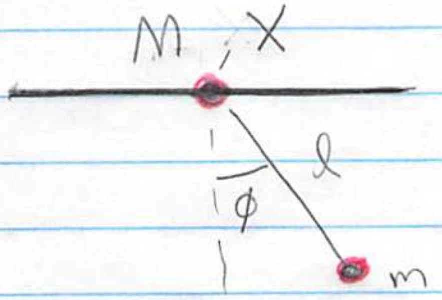
★ $p_A \equiv \frac{\partial L}{\partial \dot{q}^A}$ the momentum conjugate to the coordinate q^A

So

$$\frac{dp_A}{dt} = \frac{\partial L}{\partial q^A}$$

An Example

- Consider a pendulum on a wire. The pendulum is free to move along the wire. Determine the EOM. The rod has length l, and is light. The x-position of the top mass is $X(t)$



- Solution:

We determine $L = \frac{1}{2} m_a \vec{v}_a^{\,2} - U(\vec{x}_a)$, expressing the motion in terms of X and ϕ:

$$x_1 = X \longrightarrow \dot{x}_1 = \dot{X}$$

$$x_2 = X + l\sin\phi \longrightarrow \dot{x}_2 = \dot{X} + l\cos\phi\,\dot{\phi}$$

$$y_2 = l(1-\cos\phi) \longrightarrow \dot{y}_2 = l\sin\phi\,\dot{\phi}$$

- So

$$L = \frac{1}{2} M\dot{X}^2 + \frac{1}{2} m(\dot{X} + l\cos\phi\,\dot{\phi})^2 + \frac{1}{2} m(l\sin\theta\,\dot{\phi})^2 - mgl(1-\cos\phi)$$

$$= \frac{1}{2} M\dot{X}^2 + \frac{1}{2} m\dot{X}^2 + ml\cos\phi\,\dot{\phi}\,\dot{X} + \frac{1}{2} m l^2\dot{\phi}^2 + mgl\cos\phi + \text{const}$$

- Then

$$P_x = \frac{\partial L}{\partial \dot{x}} = M\dot{x} + m\dot{x} + ml\cos\phi\,\dot{\phi}$$ ⇐ Total X-momentum

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = ml^2\dot{\phi} + ml\cos\phi\,\dot{x}$$ ⇐ angular momentum about pivot point

- So the EOM

$$\frac{dP_x}{dt} = 0 = \frac{\partial L}{\partial x}$$ ← total momentum conservation in X

$$\frac{dP_\phi}{dt} = \frac{\partial L}{\partial \phi}$$

So

$$\frac{d}{dt}\left(ml^2\dot{\phi} + \overbrace{ml\cos\phi\,\dot{x}}^{\text{partial cancelations here}}\right) = -ml\sin\phi\,\dot{\phi}\,\dot{x} - mgl\sin\phi$$

After a pleasing cancelation:

★ $$\frac{d(ml^2\dot{\phi})}{dt} = \underbrace{-ml\cos\phi\,\ddot{x}}_{} - \underbrace{mgl\sin\phi}_{\text{Torque by gravity}}$$

★ acceleration of the whole system produces an apparant force $-m\vec{a}$, which causes a torque around the accelerating base point X.

☆ The "pleasing cancellation" of $\dot{\phi}\dot{x}$ terms had to happen:

If $\dot{x}$ is constant, that is just a different inertial frame, and the EOM should be the same in all frames. For a fixed pendulum, the EOM is

$$\frac{d}{dt}(ml^2\dot{\phi}) = -mgl\sin\phi$$

And this is the EOM if $\dot{x}$ is constant