Constraints in the Lagrangian

- Now it is easy to see this structure in the Lagrangian. The original action is

$$\vec{r} = x^A \vec{e}_A \equiv x\,\hat{e}_x + y\,\hat{e}_y + z\,\hat{e}_z$$

$$S[\vec{r}] = \int dt \; \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r})$$

- Then variation gives:

$$\delta S[\delta \vec{r}] = \int dt \left(-\frac{d}{dt} \frac{\partial L}{\partial \dot{x}^A} + \frac{\partial L}{\partial x^A} \right) \delta x^A$$

But, the δx^A are constrained at each time since

$$f(x^A, t) = 0 \Rightarrow \delta f = \frac{\partial f}{\partial x^A} \delta x^A = 0$$

- So we examine a new action $\hat{S}$

constraints

$$\hat{S}[\vec{r}, \lambda_a(t)] = \int dt \; \frac{1}{2} m \dot{\vec{r}}^2 - U(\vec{r}) + \lambda(t) f(x^A, t)$$

Varying the action gives the terms we discussed previously plus constraint related terms

$$\delta \hat{S} = \int dt \left(-\frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A} + \frac{\partial L}{\partial x^A} + \lambda(t)\frac{\partial f}{\partial x^A} \right) \delta x^A$$

$$+ \int dt \; \delta\lambda \; f(x^A)$$

- Yielding our equations of motion

$$\boxed{\begin{gathered} \frac{d}{dt}\frac{\partial L}{\partial \dot{x}^A} = \frac{\partial L}{\partial x^A} + \lambda \frac{\partial f}{\partial x^A} \\ f(x^A) = 0 \end{gathered}}$$

Lagrange EOM with constraints

- For the usual Lagrangian:

$$L = \frac{1}{2} m \dot{\vec{r}}^{\,2} - U(\vec{r})$$

This reduces to the "usual" form

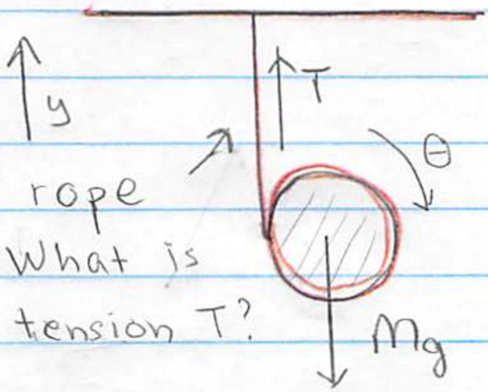
$$\frac{d}{dt}(m \dot{\vec{r}}) = -\frac{\partial U}{\partial \vec{r}} + \lambda \frac{\partial f}{\partial \vec{r}}$$

or

$$\boxed{\frac{d}{dt}(m \dot{x}_i) = -\frac{\partial U}{\partial x^i} + \lambda \frac{\partial f}{\partial x^i}}$$

Newton's Law with constraint forces

Falling Loop With Constraints:



- Disk of radius a, falling

$$y = -a\theta$$

a rolling w/out slipping constraint between the y-coordinate and θ-coordinate

- The moment of inertia $I = \int dm\, r^2 = \frac{1}{2} m a^2$

- The Lagrangian is

$$L = \underbrace{\frac{1}{2}\left(\frac{1}{2} m a^2\right)\dot{\theta}^2}_{\frac{1}{2} I \omega^2} + \underbrace{\frac{1}{2} m (a\dot{\theta})^2}_{\frac{1}{2} m v_{cm}^2} - mg(-a\theta)$$

- The EOM are:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta} \Rightarrow \frac{3}{2} m a^2 \ddot{\theta} = ga \qquad \ddot{\theta} = \frac{2}{3} ga$$

and

$$\ddot{y} = -a\ddot{\theta} = -\frac{2}{3} g$$

- Now we will solve this problem with constraints to find the tension

- Now we will compute the tension in the rope:

- The coordinates y and θ are independent in the action, but are connected by the multiplier

$$\hat{L} = \underbrace{\frac{1}{2}\left(\frac{1}{2}ma^2\right)\dot{\theta}^2}_{\frac{1}{2}I\omega^2} + \underbrace{\frac{1}{2}m\dot{y}^2}_{\frac{1}{2}mv_{cm}^2} - mgy + \lambda \underbrace{(y + a\theta)}_{\text{constraint!}}$$

- Then the EOM are:

force

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{\theta}} = \frac{\partial \hat{L}}{\partial \theta} \Rightarrow m\ddot{y} = -mg + \lambda \qquad (1)$$

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \dot{y}} = \frac{\partial \hat{L}}{\partial y} \Rightarrow \frac{1}{2}ma^2\ddot{\theta} = \lambda a \qquad (2)$$

torque

$$0 = \frac{\partial \hat{L}}{\partial \lambda} \Rightarrow y + \theta a = 0 \qquad (3)$$

$\leftarrow$ constraint

So using $\ddot{y} = -a\ddot{\theta}$ we solve the (linear) equations for $\ddot{y}$ and λ finding

$$\ddot{y} = -\frac{2}{3}g$$

$\leftarrow$ Same acceleration as before but now we get the force of constraint λ

$$\lambda = g\frac{m}{3}$$

We can interpret λ as the force of tension by looking at (1)

$$T = \lambda = g\frac{m}{3}$$

upward force

$$ma_y = -mg + \lambda$$