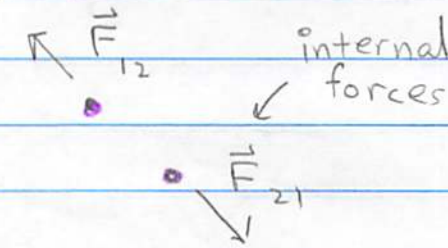
Momentum of a System of Particles

Newton:

$$\frac{d\vec{p}_a}{dt} = \vec{F}_a$$

with $\vec{p}_a = m_a \vec{v}_a$

- Now divide the forces up into external forces and internal ones:



$$\vec{F}_a = \vec{F}_a^{\,ext} + \sum_b \vec{F}_{ab}$$

Force on a-th particle by b-th

- Summing over particles:

$$\frac{d\vec{P}_{Tot}}{dt} = \sum_a \vec{F}_a^{\,ext} + \cancel{\sum_{ab} \vec{F}_{ab}} \xrightarrow{} 0$$

Equal and opposite: Newtons 3rd Law

$\vec{F}_{ab} = -\vec{F}_{ba}$

$$\boxed{\frac{d\vec{P}_{TOT}}{dt} = \vec{F}^{\,ext}_{TOT}}$$

Total external force

where

$$\vec{P}_{Tot} = \sum_a m_a \vec{v}_a = \text{Total momentum}$$

- Define the center of mass

$\bigstar$ $R_{cm} \equiv \sum m_a \vec{r}_a / M_{TOT}$ and so:

$\bigstar$ $\vec{V}_{cm} = \sum_a m_a \vec{v}_a / M_{TOT}$

So $\boxed{P_{TOT} \equiv M_{TOT} \vec{V}_{cm}}$

Angular Momentum

the EOM

$$\frac{d\vec{p}_a}{dt} = \vec{F}_a$$

- Taking the cross product:

$$\vec{r}_a \times \frac{d\vec{p}_a}{dt} = \vec{r} \times \vec{F}$$

Now:

since $\vec{p}_a = m\vec{v}_a$

$$\frac{d}{dt} \vec{r}_a \times \vec{p}_a = \cancel{v_a \times \vec{p}_a} + \vec{r}_a \times \frac{d\vec{p}_a}{dt}$$

- So the angular momentum changes as:

$$\frac{d\vec{l}_a}{dt} = \vec{r}_a \times \vec{F}_a \qquad \vec{l}_a \equiv \vec{r}_a \times \vec{p}_a$$

- Summing over the particles, and again writing

$$\vec F_a = \vec F_a^{\,ext} + \vec F_{ab}$$

We have

$$\frac{d\vec L_{TOT}}{dt} = \underbrace{\sum_a \vec r_a \times \vec F_a^{\,ext}}_{\substack{\text{external} \\ \text{torques on} \\ \text{the system}}} + \underbrace{\sum_{ab} \vec r_a \times \vec F_{ab}}_{\substack{\text{internal torque} \\ \vec \tau_{int}}}$$

with

$$\vec L_{TOT} = \sum_a \vec \ell_a \equiv \text{total angular momentum}$$

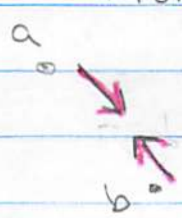
- Now the internal torques cancel (for a large variety of forces) since

$$\vec\tau_{int} = \sum_{ab} \vec r_a \times \vec F_{ab} = \frac{1}{2}\sum_{ab}\left(\vec r_a \times \vec F_{ab} + \vec r_b \times \underbrace{\vec F_{ba}}_{= -\vec F_{ab}}\right)$$

$$= \frac{1}{2}\sum_{ab}(\vec r_a - \vec r_b) \times \vec F_{ab}$$

Now typically the forces are directional

$$\vec F_{ab} \propto \vec r_a - \vec r_b \quad \text{like gravity}$$



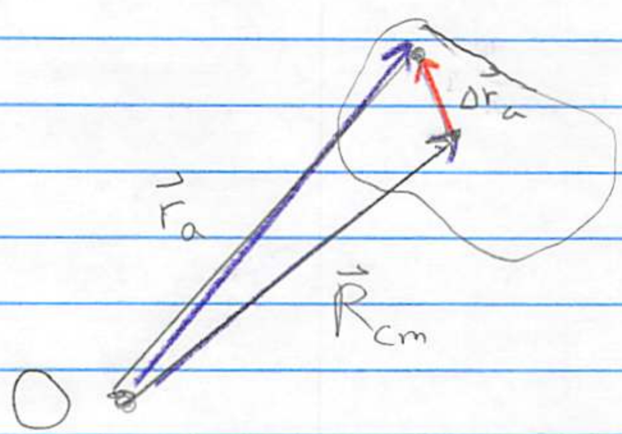
and then the internal Torques vanish, being of the form

$$(\vec{r}_a - \vec{r}_b) \times (\vec{r}_a - \vec{r}_b) = 0$$

And then

$$\boxed{\frac{d\vec{L}_{TOT}}{dt} = \vec{\tau}^{\,ext}_{TOT}}$$

- There can be modifications to this when spin is involved, and $\vec{F}_{ab} \neq$ porptiondl to $\vec{r}_a - \vec{r}_b$
- We have worked with a specific origin, O. Let us pause to determine the angular momentum around the CM:



$$\vec{r}_a = \vec{R}_{cm} + \Delta\vec{r}_a$$

So

$$\vec{L}_O = \sum_a (\vec{R}_{cm} + \Delta\vec{r}_a) \times \vec{p}_a$$

Yielding:

$$\boxed{\vec{L}_O = \underbrace{\vec{R}_{cm} \times \vec{P}_{TOT}}_{\substack{\text{angular} \\ \text{momentum} \\ \text{of CM} \\ \text{around } O}} + \sum_a \underbrace{\Delta\vec{r}_a \times \vec{p}_a}_{\substack{\text{angular} \\ \text{momentum} \\ \text{about the CM}}}}$$

$$\vec{P}_{TOT} = \sum \vec{p}_a$$

Energy

- Take a single particle

$$\frac{d\vec{p}}{dt} = \vec{F}$$

- Multiply by $\vec{v}$

$$\vec{v} \cdot \frac{d\vec{p}}{dt} = \vec{F} \cdot \vec{v}$$

- Use $\vec{p} = m\vec{v}$ and then

$$\frac{d}{dt}\left(\frac{1}{2} m v^2\right) = \vec{v} \cdot m\vec{v}$$

- To find after integration from $t_i \to t_f$

$$\frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 = \underbrace{\int_{x_i}^{x_f} \vec{F} \cdot d\vec{x}}_{W_{12}}$$

- For $\vec{F}$ conservative:

$$\vec{F} = -\frac{\partial U(x)}{\partial \vec{x}}$$

And

$$W_{if} = -\left(U(x_f) - U(x_i)\right)$$

- Yielding $T \equiv \frac{1}{2} m \dot{x}^2$

$$T(\dot{x}_f) + U(x_f) = T(\dot{x}_f) + U(x_i)$$

So energy is conserved. In general (see Tong) (or Goldstein)

$$E = \underbrace{\sum_a \frac{1}{2} m \vec{v}_a^{\,2}}_{\text{Kinetic}} + \underbrace{U^{ext}(x_a) + \frac{1}{2} \sum_{\substack{ab \\ a \neq b}} U(|\vec{r}_a - \vec{r}_a|)}_{\text{potential}}$$

- A short excercise also shows that if we write

- $\vec{v}_a = \vec{V}_{cm} + \Delta \vec{v}_a$ ← velocity relative to cm.

Then show for yourself that:

- $T = \frac{1}{2} M_{TOT} V_{cm}^2 + \frac{1}{2} \sum_a m \Delta v_a^2$

↑ Kinetic energy of cm

↑ internal kinetic energy.