1 Basic Mechanics

1.1 Newtonian mechanics a brief review

Momentum and Center of Mass

- Newton's equations of motion for a system of particles reads

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a \tag{1.1}$$

where $a = 1 \ldots N$ labels the particles. Here $\boldsymbol{p}_a = m_a \boldsymbol{v}_a$. We usually divide up the forces on the a-the particle into external forces acting on the system from outside, and internal forces acting between pairs of particles:

$$\boldsymbol{F}_a = \underbrace{\boldsymbol{F}_a^{\rm ext}}_{\text{external forces}} + \underbrace{\sum_{b\neq a} \boldsymbol{F}_{ab}}_{\text{internal forces}} . \tag{1.2}$$

Here

$$\boldsymbol{F}_{ab} \equiv \text{ Force on particle } a \text{ by } b\,, \tag{1.3}$$

and of course we have Newton's equal and opposite rule

$$\boldsymbol{F}_{ab} = -\boldsymbol{F}_{ba}\,. \tag{1.4}$$

- Summing over the particles we find (after using Eq. (1.4)) that the internal forces cancel and the total change in momentum per time is the sum of external forces

$$\frac{d\boldsymbol{P}_{\rm tot}}{dt} = \boldsymbol{F}_{\rm tot}^{\rm ext} \tag{1.5}$$

where $\boldsymbol{P}_{\rm tot} = \sum_a \boldsymbol{p}_a$ and $\boldsymbol{F}^{\rm ext}_{\rm tot} = \sum_a \boldsymbol{F}_a^{\rm ext}$. If there are no external forces then $\boldsymbol{P}_{\rm tot}$ is constant

- The velocity of the center of mass is

$$\boldsymbol{v}_{\rm cm} = \frac{\boldsymbol{P}_{\rm tot}}{M_{\rm tot}} = \frac{1}{M_{\rm tot}} \sum_a m_a \boldsymbol{v}_a\,. \tag{1.6}$$

The position of the center of mass (relative to an origin O) is

$$\boldsymbol{R}_{\rm cm} = \frac{1}{M_{\rm tot}} \sum_a m_a \boldsymbol{r}_a\,. \tag{1.7}$$

Angular momentum:

- Angular momentum is defined with respect to a specific origin O (i.e. $\boldsymbol{r}_a$ depends on O) which is not normally notated

$$\boldsymbol{\ell}_{a,O} \equiv \boldsymbol{\ell}_a \equiv \boldsymbol{r}_a \times \boldsymbol{p}_a\,. \tag{1.8}$$

It evolves as

$$\frac{d\boldsymbol{\ell}}{dt} = \boldsymbol{r}_a \times \boldsymbol{F}_a \tag{1.9}$$

- The total angular momentum $\boldsymbol{L}_{\rm tot} = \sum_a \boldsymbol{\ell}_a$ changes due to the total *external* torque

$$\frac{d\boldsymbol{L}_{\rm tot}}{dt} = \boldsymbol{\tau}^{\rm ext}_{\rm tot} \, , \tag{1.10}$$

where $\boldsymbol{\tau}^{\rm ext}_{\rm tot} = \sum_a \boldsymbol{r}_a \times \boldsymbol{F}^{\rm ext}_a$ were we have generally assumed that the internal forces are radially directed $\boldsymbol{F}_{ab} \propto (\boldsymbol{r}_a - \boldsymbol{r}_b)$

- The angular momentum depends on the origin O. Writing the position of the particle relative to the center of mass as $\Delta\boldsymbol{r}_a$, i.e.

$$\boldsymbol{r}_a = \boldsymbol{R}_{\rm cm} + \Delta\boldsymbol{r}_a \, , \tag{1.11}$$

the angular momentum of the system about O is

$$\boldsymbol{L}_O = \underbrace{\boldsymbol{R}_{\rm cm} \times \boldsymbol{P}_{\rm tot}}_{\text{Ang-mom of center of mass about } O} + \underbrace{\sum_a \Delta\boldsymbol{r}_a \times \boldsymbol{p}_a}_{\text{Ang-mom about the cm}} \, . \tag{1.12}$$

Energy

- Energy conservation is derived by taking the dot product of $\boldsymbol{v}$ with $d\boldsymbol{p}/dt$. We find that the change in kinetic energy (on the a-the particle) equals the work done (on the a-particle).

$$\left.\frac{1}{2} m_a v_a^2(t)\right|_{t_1}^{t_2} = W_a \tag{1.13}$$

where the work is

$$W_a = \int_{\boldsymbol{r}_a(t_1)}^{\boldsymbol{r}_a(t_2)} \boldsymbol{F}_a \cdot {\rm d}\boldsymbol{r}_a \tag{1.14}$$

- *Potential Energy.* For conservative forces the force can be written as (minus) the gradient of a scalar function which we call the potential energy

$$\boldsymbol{F}_a = -\nabla_{\boldsymbol{r}_a} U \tag{1.15}$$

Consider the potential energy U_{12} between particle 1 and 2. Since the force is equal and opposite

$$\boldsymbol{F}_{12} = -\nabla_{\boldsymbol{r}_1} U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = +\nabla_{\boldsymbol{r}_2} U_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\boldsymbol{F}_{21} \tag{1.16}$$

and this is used to conclude that interaction potential between two particles is of the form

$$U^{\rm int}_{12} = U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{1.17}$$

Typically we divide up the potential into an external potential and the internal ones

$$U(\boldsymbol{r}_a) = U^{\rm ext}(\boldsymbol{r}_a) + \frac{1}{2} \sum_{ab, a\neq b} U^{\rm int}_{ab}(\boldsymbol{r}_a, \boldsymbol{r}_b) \tag{1.18}$$

The sum over the internal potentials comes with a factor of a half because the energy between particle-1 and particle-2 is counted twice in the sum, e.g. for just two particles

$$U^{\rm int}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2) = \frac{1}{2} \left(U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) + U(|\boldsymbol{r}_2 - \boldsymbol{r}_1|) \right) \, . \tag{1.19}$$

- *Energy.* The total energy is

$$E = \sum_a \frac{1}{2} m_a v_a^2 + U^{\rm ext}(\boldsymbol{r}_a) + \frac{1}{2} \sum_{ab, a\neq b} U^{\rm int}_{ab}(\boldsymbol{r}_a, \boldsymbol{r}_b) \tag{1.20}$$

  and is constant if there are no non-conservative forces.
  If there are non-conservative forces then

$$E(t_2) - E(t_1) = W_{\rm NC} \tag{1.21}$$

  where the work done by the non-conservative forces is $W_{NC} = \sum_a \int \boldsymbol{F}_a^{NC} \cdot {\rm d}\boldsymbol{r}_a$

- It is convenient to measure velocities relative to the center of mass

$$\boldsymbol{v}_a = \boldsymbol{v}_{\rm cm} + \Delta \boldsymbol{v}_a \tag{1.22}$$

  where $\Delta \boldsymbol{v}_a = \dot{\Delta} \boldsymbol{r}_a$, then the kinetic energy

$$K = \underbrace{\frac{1}{2} M_{\rm tot} v_{\rm cm}^2}_{\text{KE of center-mass}} + \underbrace{\sum_a \frac{1}{2} m_a \Delta v_a^2}_{\text{KE relative to center-mass}} \tag{1.23}$$

Galilean invariance:

- Consider newtons laws then for an isolated system of particles

$$\frac{d\boldsymbol{p}_a}{dt} = \boldsymbol{F}_a \tag{1.24}$$

where $\boldsymbol{F}_a = -\nabla_{\boldsymbol{r}_a} U$ with

$$U = \frac{1}{2} \sum_{ab, a\neq b} U^{\rm int}_{ab}(|\boldsymbol{r}_a - \boldsymbol{r}_b|) \tag{1.25}$$

Here the space-time coordinates are measured by an observer O with origin.

Then consider an observer O' moving with *constant* velocity $-\boldsymbol{u}$ relative to O. The "new" coordinates (those measured by O') are related to the old coordinates via a Galilean boost

$$\boldsymbol{r}_a \to \boldsymbol{r}'_a = \boldsymbol{r}_a + \boldsymbol{u} t \tag{1.26}$$

$$t \to t' = t \tag{1.27}$$

The potential which only depends on $\boldsymbol{r}_a - \boldsymbol{r}_b$ is independent of the shift. The observer measures

$$\boldsymbol{v}_a \to \boldsymbol{v}'_a = \boldsymbol{v}_a + \boldsymbol{u} \tag{1.28}$$

$$\boldsymbol{p}_a \to \boldsymbol{p}'_a = \boldsymbol{p}_a + m_a \boldsymbol{u} \tag{1.29}$$

The equations of motion for observer O' are unchanged

$$\frac{d\boldsymbol{p}'_a}{dt'} = \boldsymbol{F}'_a \qquad \boldsymbol{F}' \equiv \nabla_{\boldsymbol{r}'} U(|\boldsymbol{r}'_a - \boldsymbol{r}'_b|) \tag{1.30}$$