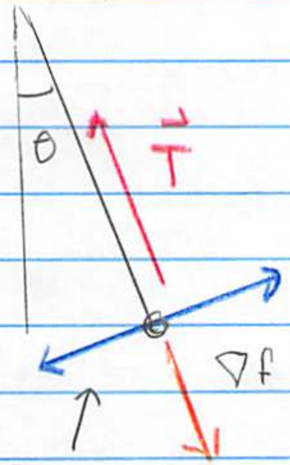
Mechanics and Constraints

Consider a pendulum. x, y are constrained:



$$f(x,y) = x^2 + y^2 - \ell^2 = 0 \qquad (1)$$

$$x = \ell \sin\theta$$
$$y = -\ell \cos\theta$$

∇f is $\perp$ to direction of variation

allowed variation of x and y
(lines of constant $f(x,y)$)

(2) $ma^x = T^x$ $\qquad\qquad$ $\vec{T}$ = force of constraint

(3) $ma^y = T^y - mg$ $\qquad\qquad$ $= -T_0 (\sin\theta, -\cos\theta)$

But how did you know this?

- We require that the forces of constraint have:

$$\boxed{\vec{T} \cdot \delta\vec{r} = 0}$$

- The constraint forces do no work under variation of the coordinates. Now we can solve (1), (2), (3) for a^x, a^y, and magnitude T_0

- Now $\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y = 0 \Rightarrow \boxed{\vec{\nabla} f \cdot \delta\vec{r} = 0}$

- So we may take

$$\vec{T} = \lambda \vec{\nabla f}$$

Then for this probem

$$\vec{T} = \lambda\ (2x, 2y)$$

$$= 2\ell\lambda \left(\frac{x}{\ell}, \frac{y}{\ell}\right) = \underbrace{2\ell\lambda}^{-T_0} (\sin\theta, -\cos\theta)$$

$$\lambda \equiv \frac{-T_0}{2\ell}$$

- And the EOM is:

$$\frac{d\vec{p}}{dt} = \vec{F} + \vec{T}$$

mechanics with constraints

$$\boxed{\frac{dp_i}{dt} = \vec{F} + \lambda \frac{\partial f}{\partial x^i}}$$

So

(1) $$\frac{d(mv_x)}{dt} = 2\lambda\ x$$

(3) $$x^2 + y^2 - \ell^2 = 0$$

(2) $$\frac{d(mv_y)}{dt} = 2\lambda\ y - mg$$

At each time we can solve for

a_x, a_y and λ (the magnitude of tension is $T_0 = -\lambda/2\ell$). Do it

Generalization to multi-component system:

- In general the for each constraint $f(\vec{r}_1, \vec{r}_2, \vec{r}_3 \ldots \vec{r}_N) = 0$ with N-particles

We reqire no virtual work by forces of constraint

$$\sum_a \vec{T}_a \cdot \delta r_a = 0$$

a-sums over the coordinates of the different particles

- Since

$$\delta f = \frac{\partial f}{\partial \vec{r}_a} \cdot \delta \vec{r}_a = 0$$

We may take

$$\vec{T}_a = \lambda \frac{\partial f}{\partial \vec{r}^a}$$

This is how the forces of constraint are specified in newton's laws. Lambda is proportional to the magnitude of the force of constraint

- Or

$$\boxed{T_{ai} = \lambda \frac{\partial f}{\partial x^{ai}}}$$

derivative with respect to the i-th cartesian component of particle a

force on a-th particle in the i-th direction