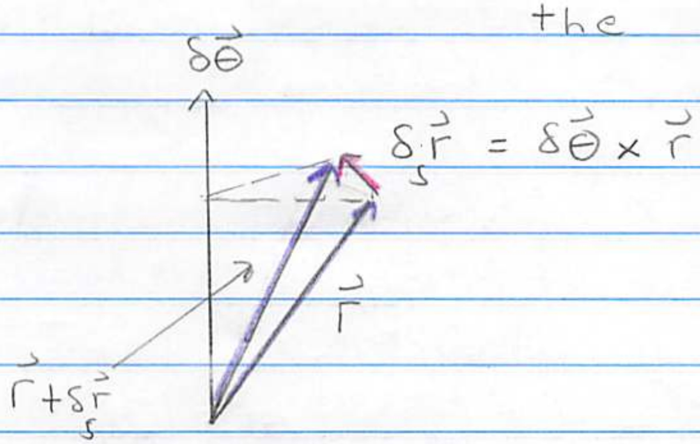
Rotations and angular momentum

- The Lagrangian

$$L = \frac{1}{2} m_1 \dot{\vec{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec{r}}_2^{\,2} - U(|\vec{r}_1 - \vec{r}_2|)$$

- If I rotate the coordinates by a small amount, the $|\vec{r}_1 - \vec{r}_2|$ is unchanged, So, the Lagrangian is unchanged by this rotation



$$\vec{r} \longrightarrow \vec{r}_a + \overbrace{\delta\vec{\theta} \times \vec{r}_a}^{\equiv\, \delta_s \vec{r}}$$

← see picture

$$L \longrightarrow L$$

small rotation around $\vec{\theta}$ direction

- So the conserved quanity is

$$Q = \sum_a \vec{p}_a \cdot \delta_s \vec{r}$$

Identity

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

$$= \sum_a \vec{p}_a \cdot \delta\vec{\theta} \times \vec{r}_a$$

$$= \sum_a \delta\vec{\theta} \cdot (\vec{r}_a \times \vec{p}_a)$$

$$Q = \delta\vec{\theta} \cdot \sum_a \vec{r}_a \times \vec{p}_a$$

So

Since $\delta\vec{\Theta}$ was arbitrary we have a conserved vector

$$\boxed{\vec{L} = \sum_a \vec{r}_a \times \vec{p}_a}$$

Energy Conservation and Homogeneity in Time

- We need a slight generalization of the Noether theorem. Previously we required that the Lagrangian be unchanged by the transformation

$$\vec{r}_a \longrightarrow \vec{r}_a + \delta_s \vec{r}_a$$

- But we know that if a Lagrangian L' differs from L by a total derivative dK/dt then L' and L will give the same EOM. So we will still consider it a symmetry if under the transformation

$$L \longrightarrow L + \frac{dK}{dt}$$

- Then

$$\star \quad \delta S[r, \delta_s r] = \int_{t_1}^{t_2} dt \, \frac{dK}{dt} = K\Big|_{t_1}^{t_2} = K(t_2) - K(t_1)$$

- Then a general onshell variation gives

$$\star\star \qquad \delta S[\underline{\vec{r}}, \delta\vec{r}\,] = \vec{p}_a \cdot \delta_s \vec{r}_a \Big|_{t_1}^{t_2}$$

This has a misprint should read $\vec{p}_a \cdot \delta\vec{r}_a$

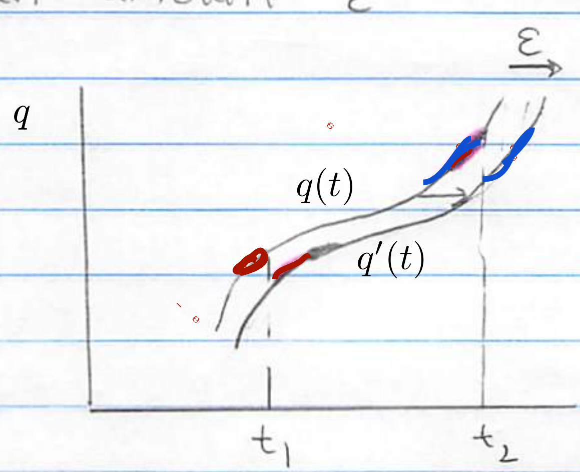
- Combining $\star$ and $\star\star$ by restricting the variation to $\delta_s r$ and setting the $\vec{r}$ onshell (i.e. demanding that $\vec{r}$ obeys the Euler-Lagrange EOM) yields a conservation law

$$Q\Big|_{t=t_2} - Q\Big|_{t=t_1} = 0$$

Where

$$\boxed{Q = \sum_a \vec{p}_a \cdot \delta_s \vec{r}_a - K}$$

- This gets used for energy conservation. Consider a path that gets shifted in time by an amount ε



Under a shift $q'(t) = q(t-\epsilon)$. The action $\int_{t_1}^{t_2} L$ is changed, but only by the contribution at the ends. The red part moves into the interval $[t_1, t_2]$ and the blue part moves out of the interval. The $K(t_2)$ and $K(t_1)$ records this change

- Now the new lagrangian is

$$L'(t) = L(t-\varepsilon) \qquad K = -\varepsilon L$$

$$= L(t) - \varepsilon \frac{dL}{dt} \quad \text{or} \quad L \to L \underbrace{- \varepsilon \frac{dL}{dt}}$$

The new coordinates are

$$\vec{r}\,'(t) = r(t-\varepsilon) \qquad \delta_s \vec{r} = -\varepsilon \dot{\vec{r}}$$

$$= \vec{r} - \varepsilon \frac{d\vec{r}}{dt} \quad \text{or} \quad \vec{r} \to \vec{r} \underbrace{- \varepsilon \frac{d\vec{r}}{dt}}$$

- So the conserved quantity is

$$Q = \sum_a p_a \cdot (-\varepsilon \dot{\vec{r}}_a) - (-\varepsilon L), \quad \text{or}$$

$$Q = -\varepsilon h \quad \text{with} \quad h = \sum_a p_a \dot{\vec{r}}_a - L$$

The minus sign is immaterial as is the constant ε, so

$$h = \vec{p}_a \cdot \delta \vec{r}_a - L \quad \text{is constant}$$

Homogeneity in time implies energy conservation