Non-Linear Oscillations Near Resonance

- First consider a damped driven oscillator near resonance

$$\ddot{x} + \eta \dot{x} + \omega_0^2 x = \frac{f_0}{m} \cos \omega t \qquad \eta \text{ small}$$

- Substituting

$$x = a \cos(-\omega t + \varphi_0) \qquad \psi \equiv -\omega t + \varphi_0$$

We find

$$(-\omega^2 + \omega_0^2)\, a \cos\psi + \overset{\omega_0\eta}{\overset{\simeq}{\omega\eta}} \sin\psi = \frac{f}{m} \cos\psi$$

- We are near resonance, $-\omega^2 + \omega_0^2 \simeq -2\omega_0(\omega - \omega_0)$, so we divide by $2\omega_0$

★★
$$\underbrace{-(\omega - \omega_0)\, a \cos\psi}_{\propto \cos\varphi_0} + \underbrace{\eta/2\, a \sin\psi}_{\propto \sin\varphi_0} = \frac{f}{2m\omega_0} \cos\omega t$$

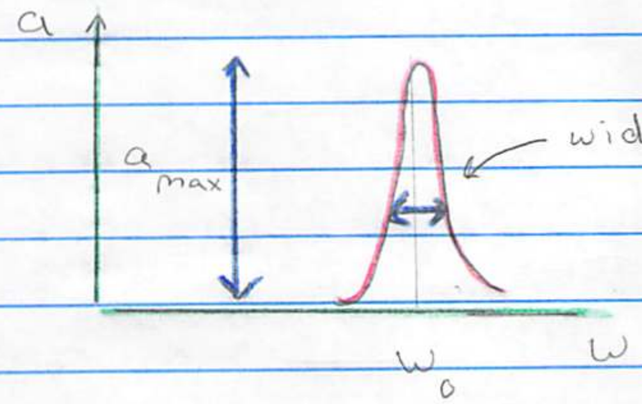
- We adjust the phase and and amplitude, $a + \varphi_0$, so ★★ works:

$$a = \left(\frac{f}{2m\omega_0}\right) \frac{1}{[(\omega - \omega_0)^2 + (\eta/2)^2]^{1/2}} \longleftarrow$$

this becomes large and non-linearites become important

$$\varphi_0 = \tan^{-1} \frac{\eta/2}{(\omega_0 - \omega)}$$

As is familiar



- The width is $\sim \eta/2$. The Q of the oscillator is $Q \equiv \omega_0/\eta$

- The maximum values is $a_{max} \equiv f_0/m\omega_0\eta$

- Non-linearities shift the frequency:

$$\Delta\omega = \frac{3}{8}\frac{\beta a^2}{\omega_0} \equiv \kappa a^2 \qquad \kappa = 3/8\, \beta/\omega_0$$

- So non-linearities will distort the line shape when

$$\Delta\omega \sim \eta/2 \qquad \text{or} \qquad \eta/2 \sim \kappa a_{max}^2$$

Or $$1 \sim \frac{\kappa}{(\eta/2)}\left(\frac{f_0}{m\omega_0\eta}\right)^2$$. Thus we will

define a dimensionless force, $\boxed{\bar{f} \equiv (\kappa/\eta/2)^{1/2}\, f/m\omega_0\eta}$
and non-linearities are important for

$$\bar{f} \sim 1$$

- Now if non linear terms are present

$$\boxed{\ddot{x} + \eta \dot{x} + \omega_0^2 x + \beta x^3 = \frac{f_0}{m} \cos \omega t}$$

- Now if $x^{(0)} = a\cos(-\omega t + \varphi_0)$ the non-linear term gives

$$\beta x^3 = \frac{3}{4}\beta a^2 \; a\cos\psi + \cos 3\psi \text{ term}$$

$$= 2\omega_0 \Delta\omega \cos\psi + \text{neglected } \cos 3\psi \text{ term}$$

Where $\Delta\omega_0(a) = \frac{3}{8}\frac{\beta a^2}{\omega_0} = K a^2 \equiv$ non linear frequency shift.

- The presence of βx^3 (divided by $2\omega_0$) thus adds $\Delta\omega \cos\psi$ to Eq ★★ on previous pages leading to

$$\xleftrightarrow{\text{adjust } a \text{ and } \varphi_0 \text{ so it holds}}$$

$$-(\omega - \omega_0(a))\, a\cos\psi + \eta/2 \, a\sin\psi = \frac{f}{2m\omega_0}\cos\omega t$$

Where $\boxed{\omega_0(a) \equiv \omega_0 + \Delta\omega_0(a) = \omega_0 + K a^2}$

- Now we follow the same procedure as the damped SHO. Compare to ★★ two pages back

So the amplitude is now satisfying

$$\boxed{a^2 = \left(\frac{f}{2m\omega_0}\right)^2 \frac{1}{[(\omega-\omega_0(a))^2 + (\eta/2)^2]}}$$

$$\tan\varphi_0 = \frac{\eta/2}{\omega_0(a)-\omega}$$

- This is a cubic equation for a^2 which reads

$$a^2(\omega-\omega_0-\kappa a^2)^2 + (\eta/2)^2 a^2 = (f/2m\omega_0)^2$$

- By appropriate units $\eta/2 = 1$, $\kappa = 1$ then

$$\bar{a}^2 \equiv \left(\frac{\kappa}{\eta/2}\right) a^2, \qquad \bar{f}^2 = \left(\frac{\kappa}{\eta/2}\right)\left(\frac{f_0}{m\omega_0\eta}\right)^2,$$

$$\xi = (\omega-\omega_0)/(\eta/2)$$

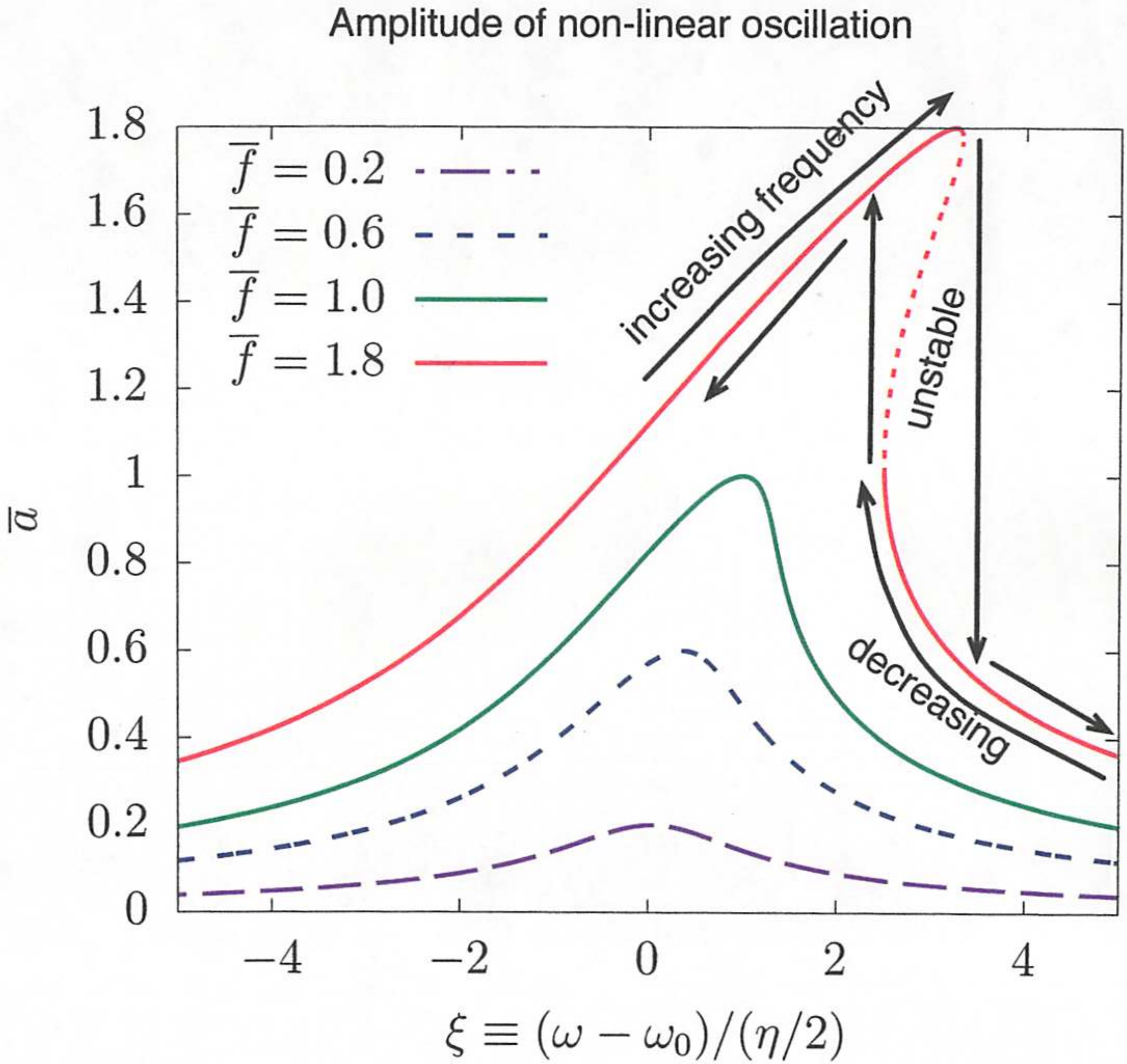
nonlinear forces are important if the dimensionless variables, barred a and f, are of order unity

which are dimensionless amplitude, force, frequency

$$\bar{a}^2(\xi-\bar{a}^2)^2 + \bar{a}^2 = \bar{f}^2$$

- Its a cubic equation you can ask mathematica to find the roots and make a graph of $\bar{a}(\xi)$ for various $\bar{f}$

See picture


Amplitude of non-linear oscillation
$\overline{f} = 0.2$
$\overline{f} = 0.6$
$\overline{f} = 1.0$
$\overline{f} = 1.8$
increasing frequency
unstable
decreasing
$\overline{a}$
$\xi \equiv (\omega - \omega_0)/(\eta/2)$
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
−4
−2
0
2
4

See Picture

- For small $\bar{f} \ll 1$, the amplitude is small, and the line shape is a simple Lorentzian

$$\bar{a}^2 = \frac{\bar{f}^2}{\xi^2 + 1}$$

There is only one real root of the cubic equation

- For larger $\bar{f}$ the line shape is distorted. For $\bar{f} \gtrsim 1$ there are three real roots causing the curve to bend over

- In the "bend-over" case as the frequency is increased, the amplitude will follow the upper branch to the tip. It will then jump to the lower branch (The middle branch is unstable -- not shown) and follow the lower branch.

- If the frequency is subsequently decreased the system will follow the lower branch until jumping to the upper branch at the lower tip.