Secular Perturbation Theory

- Trying to solve

$$\ddot{x} + \omega_0^2 x = \underbrace{-\beta x^3}_{\text{small}} \equiv f_{ind}/m$$

Now we try

$$x(t) = x^{(0)} + x^{(1)} + x^{(2)}$$

But we try a more general form for the zeroth order solution:

$$x^{(0)} = \mathrm{Re}\left[A(t)\, e^{-i\omega_0 t}\right]$$

$$= a(t)\cos(-\omega_0 t + \varphi(t))$$

- Now the amplitude and phase are slow functions of time, adjusted to remove secular divergences

$$da/dt \ll a\omega_0 \qquad d\varphi/dt \ll \omega_0$$

- Note define $\boxed{\psi \equiv -\omega_0 t + \varphi(t)}$;

$$\frac{d^2 x^{(0)}}{dt} = -\omega_0^2 X^{(0)} + \overbrace{2\omega_0\dot{\varphi}\, a\cos\psi - 2\omega_0\dot{a}\sin\psi}^{\text{extra}}$$

$$+ \text{ neglectable second derivs of } a \text{ and } \varphi$$

- As before

$$\frac{f_{ind}}{m} \simeq -\beta (x^{(0)})^3 = -\frac{\beta a^3}{4} \cos(3\psi) - \frac{3}{4}\beta a^3 \cos(\psi)$$

- Substituting $x = x^{(0)} + x^{(1)}$ into $\ddot{x} + \omega_0^2 x = f_{ind}/m$ we find:

$$\frac{d^2 x^{(1)}}{dt^2} + \omega_0^2 x^{(1)} = \underbrace{-\frac{\beta a^3}{4} \cos(3\psi)}_{f^{(1)}} + \left(-\frac{3}{4}\beta a^2 - 2\omega_0 \dot{\varphi}\right) a \cos\psi$$

$$+ (-2\omega_0 \dot{a}) \sin\psi$$

- The $\cos\psi$ and $\sin\psi$ terms are secular. We adjust $\dot{\varphi}$ and $\dot{a}$ so these do not appear

So we find

$$-2\omega_0 \dot{a} = 0 \longrightarrow \boxed{a = \text{Constant}}$$

$$-\frac{3}{4}\beta a^3 - 2\omega_0 \dot{\varphi} = 0 \longrightarrow \boxed{\varphi = -\Delta\omega\, t + \varphi_0 \quad \text{with } \Delta\omega = \frac{3}{8}\left(\frac{\beta a^2}{\omega_0^2}\right)\omega_0}$$

- The remaining $f^{(1)}$ term procedes as before

$$x^{(1)} = -\frac{1}{32}\left(\frac{\beta a^2}{\omega_0^2}\right) a \cos(3\psi) \qquad \psi = -(\omega_0 + \Delta\omega)t + \varphi_0$$

- So finally

$$x(t) = a\cos(\omega t + \varphi_0) + x^{(1)}$$

where

$$\omega = \omega_0 + \Delta\omega = \omega_0 \left(1 + \frac{3}{8}\left(\frac{\beta a^2}{\omega_0^2}\right)\omega_0\right)$$

Comment:

- For a steady state secular perturbation theory reduces to taking $x^{(0)} = a\cos(-\omega_0 t + \varphi)$

$$a = \text{const}$$

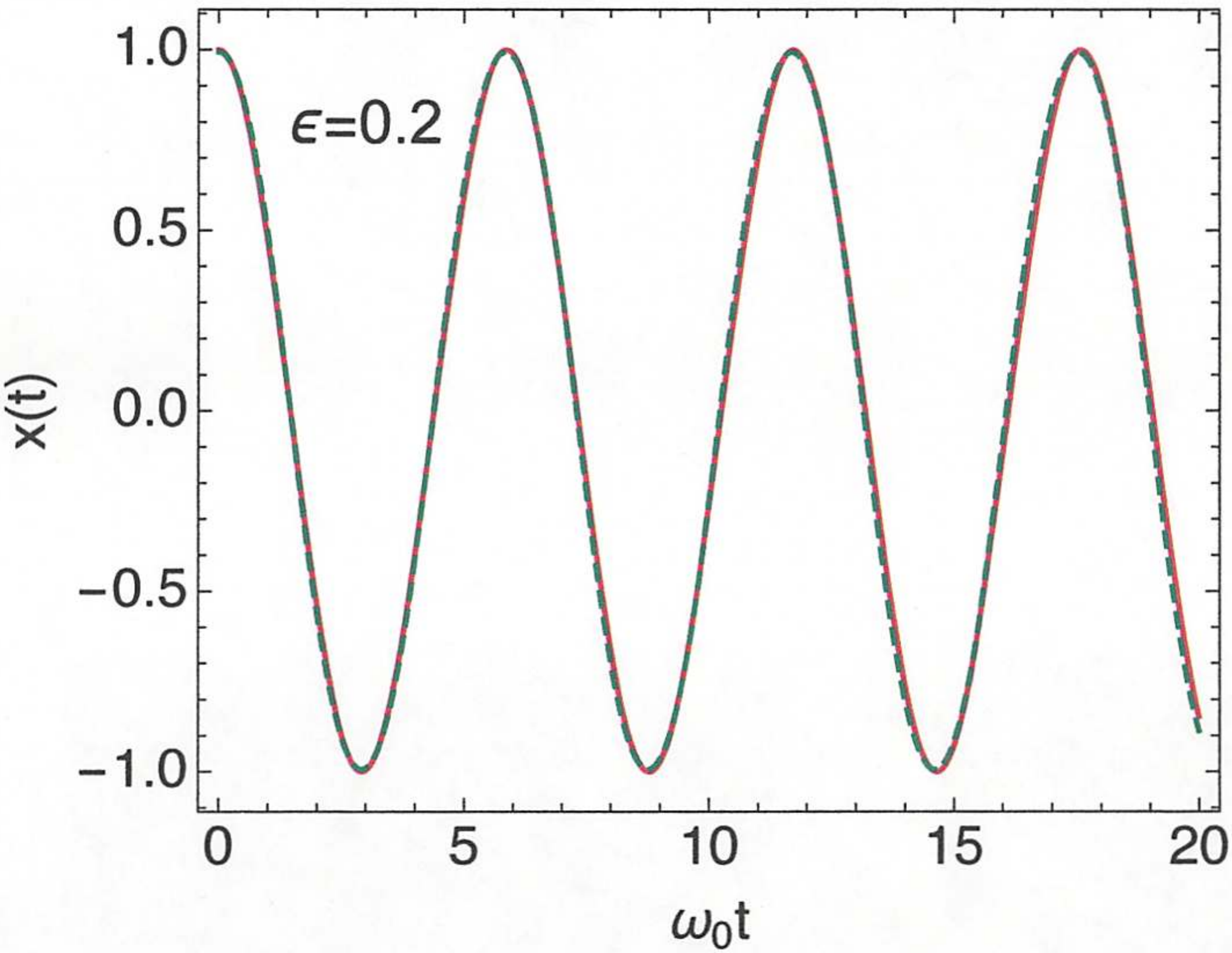
shift

$$\varphi = -\Delta\omega t + \varphi_0$$

where a and $\Delta\omega$ are adjusted at each order so that secular terms do not appear.

Non linear oscillator treating secular term as frequency shift

$$\epsilon = \frac{\beta a^2}{\omega_0^2}$$



Solid lines exact solution, dashed lines approximate