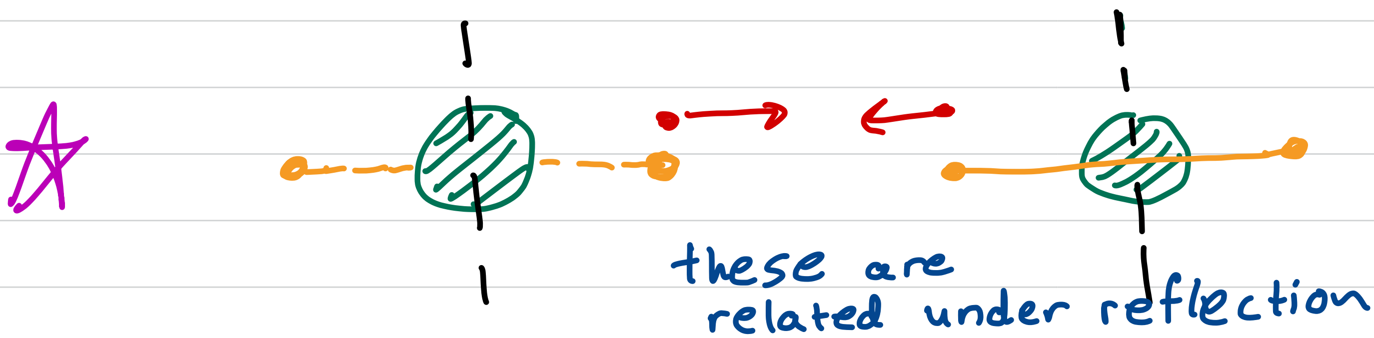
Classification of modes using symmetry



- The problem is cleary symmetric under reflections. As a result of this symmetry the normal modes will either be even or odd under reflection.

- For each deformation $\vec{X} = (x_1, x_2, x_3)$ there is a reflected deformation $\underline{\vec{X}}$

$$(x_1, x_2, x_3) \longrightarrow (\underline{x}_1, \underline{x}_2, \underline{x}_3) = (-x_3, -x_2, -x_1)$$

Thus $\vec{e}_1 = (1,0,0) \longrightarrow \underline{\vec{e}}_1 \equiv (-1,0,0)$ as shown in ☆ above

- We say a mode (or set of displacements) is even if

$$(\underline{x}_1, \underline{x}_2, \underline{x}_3) = (x_1, x_2, x_3)$$

And odd if

$$(\underline{x}_1, \underline{x}_2, \underline{x}_3) = -(x_1, x_2, x_3)$$

(for the displacements)

- It is advisable to use coordinates which reflect the symmetries of the problem. For instance the only even mode is parametrized by the coordinate q_e

$$(x_1, x_2, x_3)_{even} = q_e(-1, 0, 1)$$

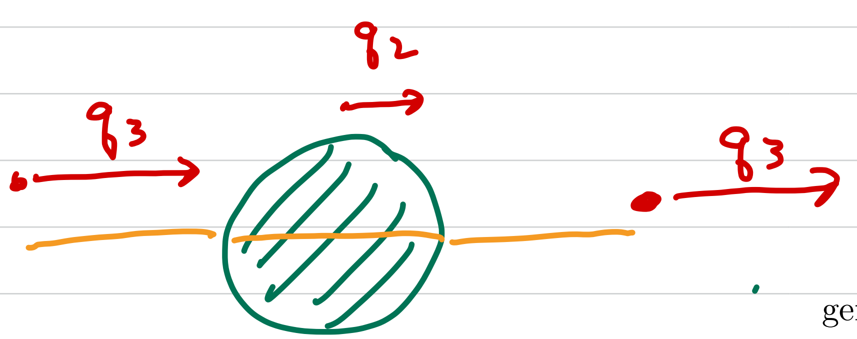
even

q_e uncoupled from other coords

Since the space of even deformations is one dimensional, it must be a normal mode, it cant "mix" with other odd modes since the problem is symmetric

- The odd modes are parametrized by two coordinates q_2 and q_3

$$(x_1, x_2, x_3)_{odd} = (q_3, q_2, q_3)$$



odd subspace spanned by two generalized coordinates

The coordinates q_2 and q_3 will be coupled, but uncoupled with q_e.

- Finally the zero modes are easy to guess and correspond to shifts in the center of mass

$$\vec{X}_o = q_{cm}(1,1,1)$$ this is odd

We want q_{cm} as one of our coordinates and require that the remaining odd modes be orthogonal to this (with respect to the weighted inner product)

$$(1\ 1\ 1)\begin{pmatrix} m & & \\ & M & \\ & & m \end{pmatrix}\begin{pmatrix} q_3 \\ q_2 \\ q_3 \end{pmatrix} = 0$$

Gives:

$$m q_3 + M q_2 + m q_3 = 0 \quad \rightarrow \quad q_2 = \frac{-2m}{M} q_3$$

- This is the requirement that the odd deformation does not change the center of mass

- This means we should have parametrized the odd deformations by q_o, q_{cm} instead of q_2, q_3

$$(q_3, q_2, q_3) = q_o \underbrace{\left(1, -\frac{2m}{M}, 1\right)}_{\text{mode}} + q_{cm}(1,1,1)$$

this is the only odd non-zero mode and must therefore be an eigenvector.

- So if we had parametrized the oscillations by coordinates q_e, q_o, q_{cm}:

$$
\begin{aligned}
(x_1, x_2, x_3) = \; & q_e(t)\,(1, 0, 1) \\
& + q_o(t)\,\left(1, -\frac{2m}{M}, 1\right) \\
& + q_{cm}(t)\,(1, 1, 1),
\end{aligned}
$$

We would have found an uncoupled oscillations problem. Here we could find all eigenmodes with symmetry and orthogonality to zero modes. In general symmetry will only reduce the number of coupled coordinates, and simplify the problem.