Ponderamotive Force

- Landau 30 / Fowler UVA

Consider a particle in a potential $m\ddot{x} = -\partial U/\partial x$. Now add an external force:

$$f(x,t) = f_0(x)\cos(-\omega t + \varphi(x))$$

$$= f_1(x)\cos\omega t + f_2(x)\sin\omega t$$

Where ω is large compared to $1/T$, $\omega \gg 1/T$, with T is a typical timescale for motion in the potentail.

- The rapidly oscillating force will generate oscillations around a smooth path

$$\boxed{x(t) = \underbrace{X(t)}_{\text{smooth path}} + \underbrace{\xi(t)}_{\text{rapid oscillations}} \leftarrow \text{assumed to be small}}$$

- You might think that $X(t)$ would forllow the potential $U(x)$, but in fact the oscillations modify the effective potential that the particle experiences

- Then for small ξ

$$m\ddot{X} + m\ddot{\xi} = -\frac{\partial U(X)}{\partial X} + \xi\frac{\partial^2 U(X)}{\partial X^2}$$

$$+ f(X,t) + \xi\frac{\partial f}{\partial X}$$

- Now ξ is small, but $\ddot{\xi}$ is not

$$\ddot{\xi} \sim \omega^2 \xi$$

Since ω is large.

- Equating the slow and fast parts

$$m\ddot{\xi} = f(X,t)$$

- And for the slow part

$$m\ddot{X} = -\frac{\partial U(X)}{\partial x} - \overline{\xi\frac{\partial^2 U(X)}{\partial x}} + \overline{\xi\frac{\partial f}{\partial x}}$$

Here we are averaging to remove any fast oscillations over a time Δt which is long compared to ω^{-1} but short compared to T

$$\omega^{-1} \ll \Delta t \ll T$$

- Now we can solve for ξ

$$\xi = -\frac{f(x,t)}{m\omega^2}$$ ← see our results for a driven oscillator

- Then

$$\overline{\xi \frac{\partial^2 U(x)}{\partial x^2}} = 0$$

sinusoidal ↗ ↖ constant

But

$$\overline{\xi \frac{\partial f}{\partial x}} = -\overline{\frac{f}{m\omega^2}\frac{\partial f}{\partial x}} = -\frac{\partial}{\partial x}\frac{\overline{f^2}}{2m\omega^2}$$

$$= -\frac{\partial}{\partial x}\frac{(f_1^2 + f_2^2)}{2m\omega^2}$$

- So the slow motion satisfies

$$\boxed{m\ddot{X} = -\frac{\partial U_{eff}}{\partial X}}$$

Where

$$\boxed{U_{eff} = U(x) + \frac{\overline{f^2(x)}}{2m\omega^2}}$$ ← the pondermotive potential

For harmonic motion the velocity is

$$|\dot{\xi}| = \left|\frac{f}{m\omega}\right|$$

We see that

$$\boxed{U_{eff} = U + \underbrace{\frac{1}{2} m \overline{\dot{\xi}^2(x)}}_{\text{average kinetic energy at point } x}}$$

Comment

- We have used the variable x and force f. We could have used a general coordinate q

$$L = \frac{1}{2} m_{eff}\, \dot{q}^2 - U(q) + \underbrace{L_{fast}(q)}_{\propto \cos\omega t}$$

Then the EOM is

$$m_{eff}\, \ddot{q} = -\frac{\partial U}{\partial q} + \frac{\partial L_{fast}}{\partial q}$$

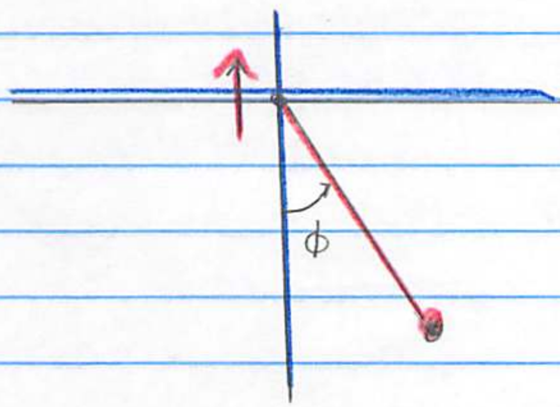
force $f_q = f_1(q)\cos\omega t + f_2(q)\sin\omega t$

The ponderamotive potential is

$$\boxed{U_{eff} = U(q) + \frac{\overline{f_q^2}}{2 m_{eff}\, \omega^2}}$$

- It can also be generalized to more coordinates

Example: Inverted pendulum with driven support



$$y = a\cos\Omega t - l\cos\phi \qquad x = l\sin\phi$$

Then

$$L = \frac{1}{2} m l^2 \dot\phi^2 - m l a \Omega \sin\Omega \sin\phi\,\dot\phi + \overbrace{\frac{1}{2} m a^2 \Omega^2 \cos^2\Omega t}^{1}$$

$$- \underbrace{mg\,(a\cos\Omega t}_{2} - l\cos\phi)$$

Neglecting total derivatives 1, 2:

$$L = \frac{1}{2} m l^2 \dot\phi^2 - m l a \Omega\, \underbrace{\sin\Omega t \sin\phi\,\dot\phi} + m g l \cos\phi$$

This term is also a total derivate

$$= -\frac{d}{dt}(\sin\Omega t\cos\phi) + \underbrace{\Omega\cos\Omega t\cos\phi}_{\text{no } \dot\phi \text{ term!}}$$

So

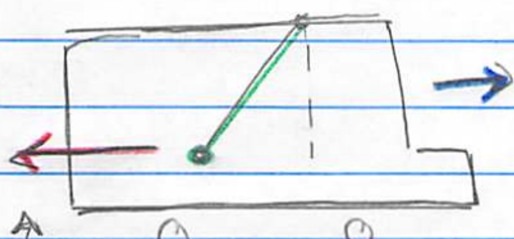
$$L = \frac{1}{2} m l^2 \dot{\phi}^2 - l\, m a \Omega^2 \cos\Omega t \cos\phi + mgl\cos\phi$$

So the EOM:

$$\underbrace{m l^2}_{m_{eff} = m l^2} \ddot{\phi} = \underbrace{+ l\, m a \Omega^2 \cos\Omega t \sin\phi}_{\text{"generalized force"} \;\equiv\; \text{torque } f_\phi} \underbrace{- mgl\sin\phi}_{-\partial U/\partial q \;\; \text{gravitational torque}}$$

- So the oscillating force is easy to interpret

If I have an accelerating bus

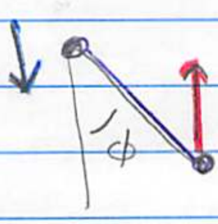


accelerating bus

$F_{eff} = -m a_{bus}$

There is an effective force $F_{eff} = -m a_{bus}$

Here at time $t=0$ we have:

$a_{support}$  $F_{eff} = -m a_{support} = m a \Omega^2 \cos\Omega t$

- And the torque it produces (by it I mean the effective force created by the accelerating support. This effective force creates the torque)

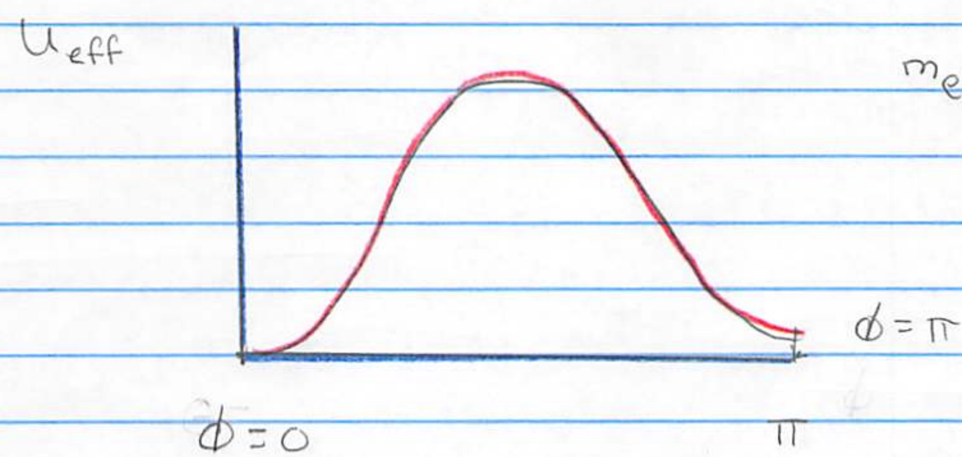
$$f_\phi = ma\Omega^2 \cos\Omega t \; l \sin\phi$$

Now we are ready to evaluate U_{eff}

$$U_{eff} = U(\phi) + \frac{\overline{f_\phi^2}}{2m_{eff}\Omega^2} \qquad \overline{\cos^2\Omega t} = 1/2$$

$$U_{eff} = -mgl\cos\phi + \frac{ma^2\Omega^2}{4}\sin^2\phi$$

- Now Lets analze the Effective potential for $ma^2\Omega^2/4 \gg mgl$. Then we neglect the $\cos\phi$ term



$$m_{eff}\ddot{\phi} = -\frac{\partial U_{eff}}{\partial\phi}$$

So we see that $\phi = \pi$ is a minimum of the potential! Let us analyze the stability incorporating the gravitational potential: Expand

$$U_{eff} = -mgl\cos\phi + ma^2\Omega^2/4 \sin^2\phi \quad \text{near } \phi = \pi$$

• Near $\phi = \pi$, $\cos\phi = -1 + \phi^2/2$, $\sin^2\phi = \phi^2$

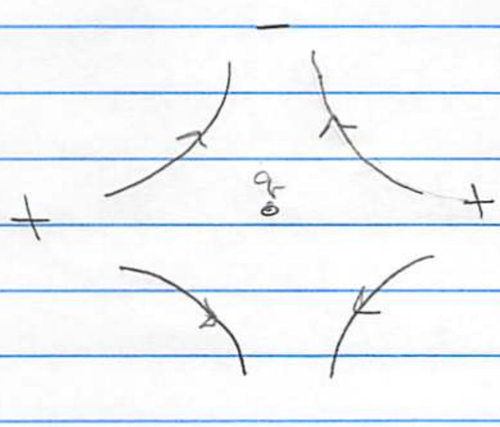
$$\frac{U}{mgl} = \left[\, 1 + \underline{\left(-1/2 + a^2\Omega^2/4gl\right)}\,\phi^2 + \ldots \right]$$

So stability at $\phi = \pi$ only happens when U is concave up. So we require the underlined term to be > 0:

$$\boxed{a^2\Omega^2 > 2gl}$$

(1) Watch video (click me)

(2) Another example of the ponderamotive force is the Paul Trap for charged ions. Recall that an electrostatic field can't be used to create a stable minimum since field lines do not end



$$\phi = \frac{k}{2}\left(x^2 + y^2 - 2z^2\right)$$

$$f \propto -q\,\nabla\phi$$

$$f \propto -x\,\hat{x} - y\,\hat{y} + 2z\,\hat{z}$$

restoring force

non-restoring force!

- By having the polarity switch signs rapidly

$$\phi = \frac{k}{2} \cos \Omega t \, (x^2 + y^2 - 2z^2)$$

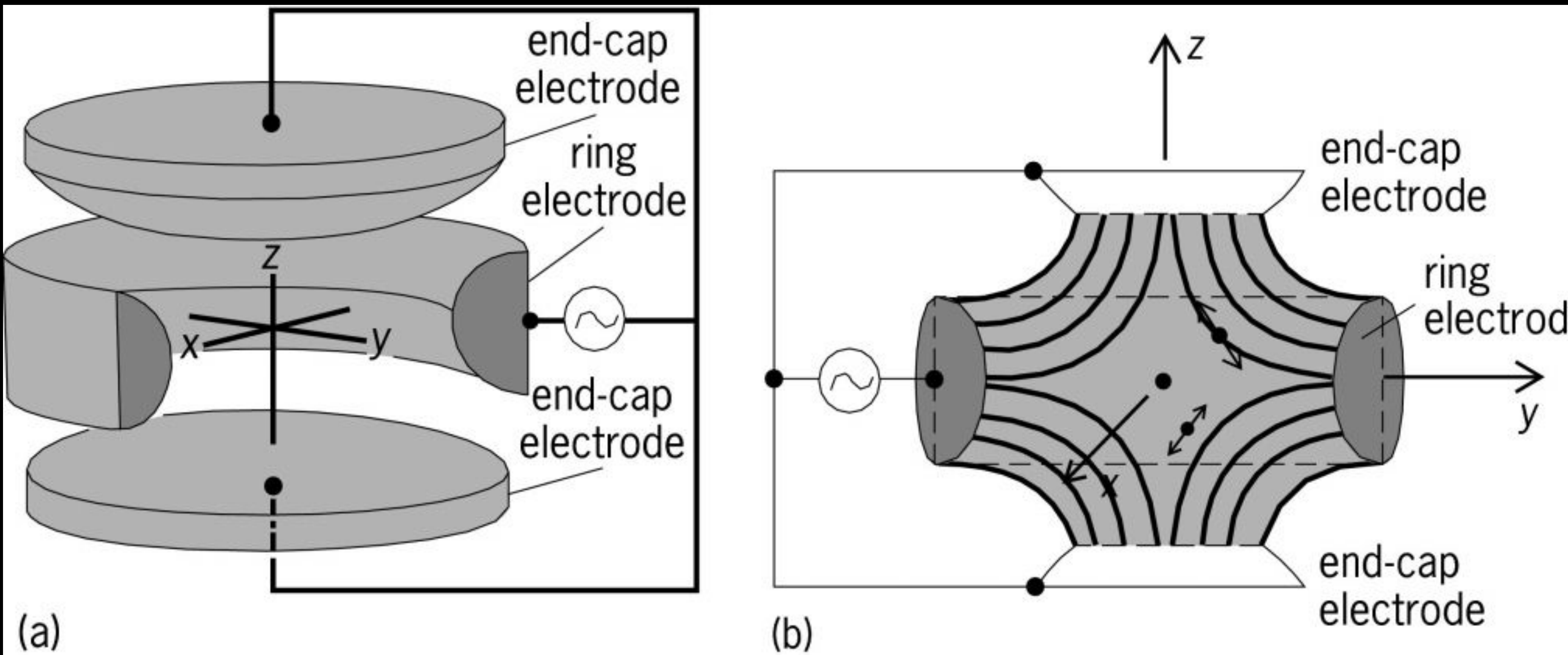
- We create a resoring force $\propto \Omega^2$

$$f_{eff} \propto -\nabla \overline{f^2} = -\nabla (x^2 + y^2 + 4z^2)$$

in all three directions

- Design and implementation of the Paul Trap earned Wolfgang Paul and Hans Dehmelt a nobel Prize in 1989

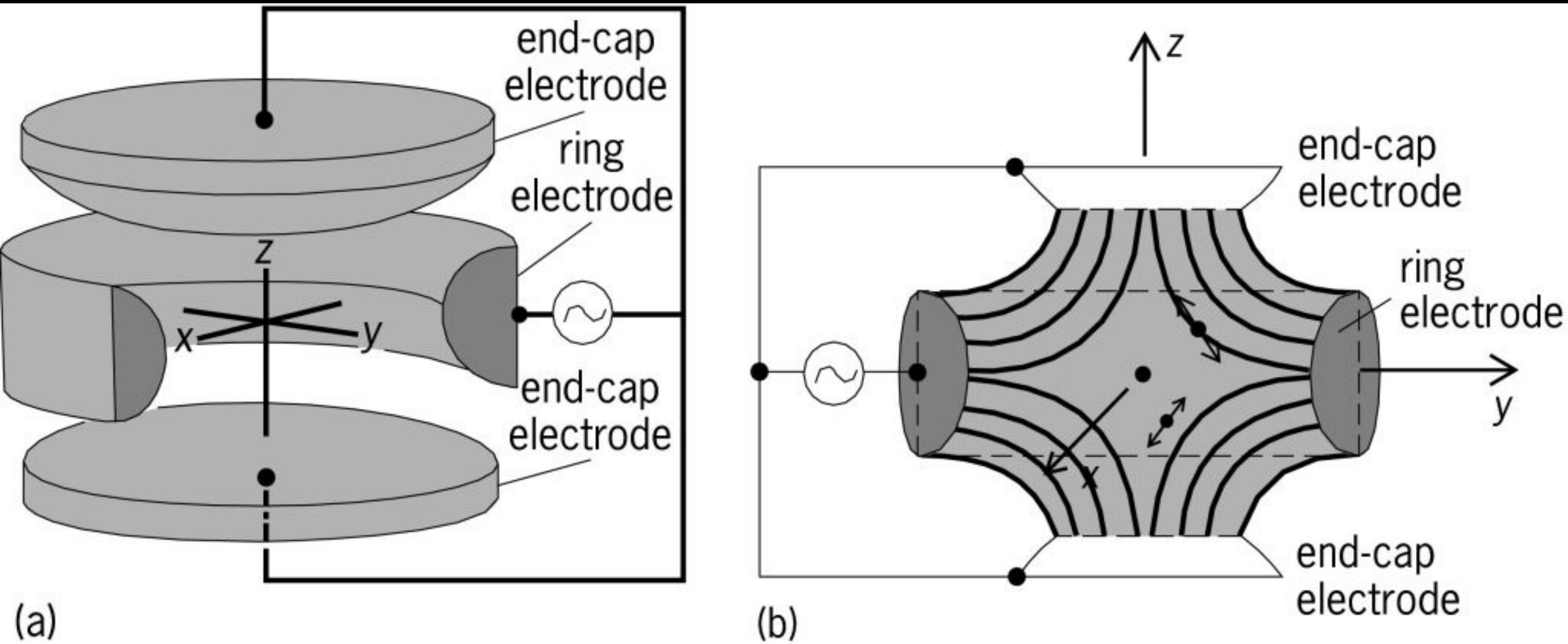
A Paul Trap for Ion Trapping



Electrostatic potential: there is no stable point

$$\phi = \frac{1}{2}A(x^2 + y^2 - 2z^2)$$

A Paul Trap for Ion Trapping



Electrostatic potential with a rapidly changing polarity:
this is stable!

$$\phi = \frac{1}{2} A \cos(\Omega t)(x^2 + y^2 - 2z^2)$$