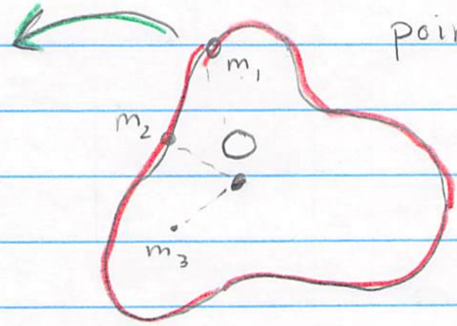
The kinetic Energy

$$T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^{\,2}$$

Body rotating about a <u>fixed</u> point O



- Now

$$\frac{d\vec{r}}{dt} = \vec{\omega} \times \vec{r}$$

So

$$T = \frac{1}{2} \int dm \; (\vec{\omega} \times \vec{r}) \cdot (\vec{\omega} \times \vec{r})$$

$|\vec{\omega} \times \vec{r}| = \omega r \sin\theta$

$$= \frac{1}{2} \int dm \; \omega^2 r^2 \underbrace{(1 - \cos^2\theta)}_{\sin^2\theta}$$

$$= \frac{1}{2} \int dm \; \omega^2 r^2 - (\vec{\omega} \cdot \vec{r})^2$$

$$= \frac{1}{2} \int dm \; \omega_a \left(r^2 \delta_{ab} - r^a r^b \right) \omega_b$$

Or

$$\boxed{T = \frac{1}{2} \omega_a I^O_{ab} \omega_b}$$

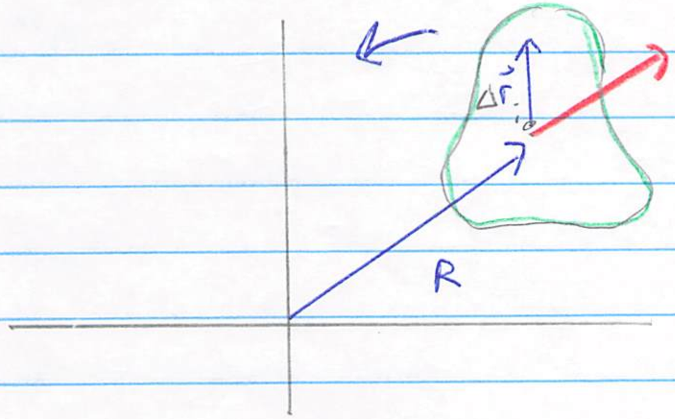
$$I^O_{ab} = \int dm \; (r^2 \delta_{ab} - r_a r_b)$$

moment of inertia tensor around the point O.

$$T = \frac{1}{2} \vec{\omega} \cdot I^O \cdot \vec{\omega}$$

$\vec{r}$ is relative to point O.

Center of Mass Motion



We have been considering a body rotating around a fixed point.

In general

position of cm

$$\vec{r}_i = \vec{R}(t) + \Delta\vec{r}_i \qquad \text{now} \quad \sum_i m_i \Delta\vec{r}_i = 0$$

this defines the CM.

$$\frac{d\vec{r}_i}{dt} = \vec{V}_{cm}(t) + \frac{d\Delta\vec{r}_i}{dt}$$

$$\frac{dr_i}{dt} = \vec{V}_{cm}(t) + \vec{\omega} \times \Delta\vec{r}_i \qquad \text{and} \quad \sum_i m_i \Delta r_i = 0$$

So

$$KE = \frac{1}{2}\sum_i m_i \frac{d\vec{r}}{dt}^2$$

$= 0$ since $m_i \Delta\vec{r}_i = 0$

$$= \frac{1}{2}\sum_i m_i \left(V_{cm}^2 + \vec{V}_{cm}\cdot(\vec{\omega}\times\Delta\vec{r}_i) + (\vec{\omega}\times\Delta\vec{r})^2\right)$$

same as before, rel. to CM.

cm

$$\boxed{T = \frac{1}{2} M_{TOT} \vec{V}_{cm}^2 + \frac{1}{2}\omega_a I_{ab}^{cm}\omega_b}$$

relative KE

The angular momentum works the same way

angular momentum about O

$$\vec{L}_O = \sum_i m_i \vec{r}_i \times \vec{v}_i$$

$$= \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i)$$

$\vec{v} = \vec{\omega} \times \vec{r}$

Use the "bac" to "abc" rule

$$a \times (\vec{b} \times \vec{c}) = \vec{b}\,(\vec{a} \cdot \vec{c}) - (\vec{a} \cdot \vec{b})\,\vec{c}$$

So

$$\vec{L} = \sum_i m_i\, \vec{\omega}\,(\vec{r}_i \cdot \vec{r}_i) - (\vec{r} \cdot \vec{\omega})\,\vec{r}$$

$$L_a = \sum_i m_i\, (\vec{r}^{\,2}\, \delta_{ab} - r_a r_b)\, \omega^b$$

Or

$$\boxed{L_a = I_{ab}\, \omega^b}$$

$$\vec{L} = I \cdot \vec{\omega}$$

Similarly if the center of mass is moving

$$(\vec{L}_O)_a = (\vec{R}_{cm} \times (M\vec{v}))_a + I^{cm}_{ab}\,\omega^b$$

a-th component of angular momentum around O

The upshot:

In many cases, e.g. tumbling of a mechanics book in free fall, the wobble of the earth's etc, we can neglect the effect of gravity and the CM motion. It decouples from the free rotations of the body

Computing Moment of Inertia Tensor

- In general need to do 6-integrals

$$I = \int d^3r\, \rho(\vec{r}) \begin{pmatrix} y^2+z^2 & xy & xz \\ xy & x^2+z^2 & yz \\ xz & yz & x^2+y^2 \end{pmatrix}$$

I like to think of this as mass weighted average of the coordinates, so $I_{xy} \equiv M\langle xy\rangle \equiv \int dV\, \rho(r)\, xy$.

Since I_{ab} is a symmetric matrix we can always by fixed rotation of coordinates diagonalize this matrix by

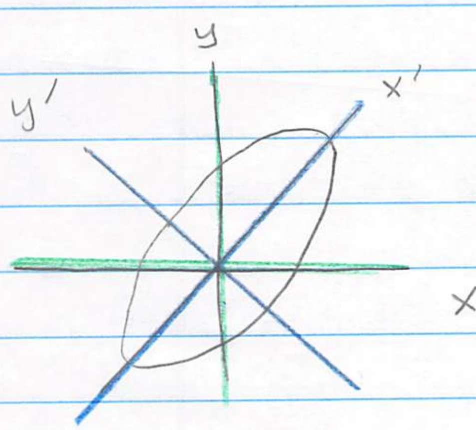
$$I \to I' = O I O^T$$

fixed rotation

I after rotation O

$$I' = \begin{pmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{pmatrix}$$

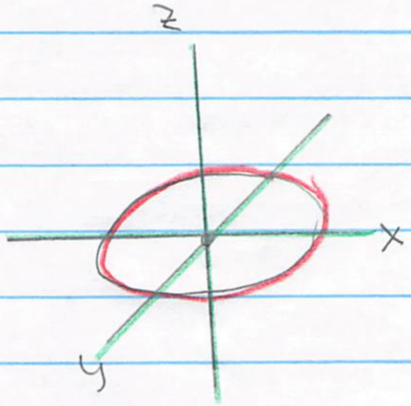
- This rotated coordinate system is known as the principal axes. Usually one can guess the princpal axes, e.g by symmetry



you could use the xy body coordinates here, but that would by crazy

The principal axes are along x' and y'. And this is a much more natural body coordinate system than the x, y system

- Take a disk of radius R



(1) $I_{xy} = M\langle xy\rangle = 0$

Since the body is symmetic under $x \to -x$ this integral vanishes

(2) $I_{xz} = M\langle xz\rangle = 0 = I_{yz}$

$z = 0$ on disk

(3) $I_{xx} = M\langle y^2 + \cancel{z^2}\rangle = M\langle y^2\rangle$

$I_{yy} = M\langle x^2 + \cancel{z^2}\rangle = M\langle x^2\rangle$

same by symmetry of $x, \& y$

$I_{zz} = M\langle x^2 + y^2\rangle = I_{xx} + I_{yy}$

I_{zz} is easy $\quad I_{zz} = \frac{1}{2} mR^2 = 2I_{xx} = 2I_{yy}$

Parallel axis theorem

- Suppose we know the moment of inertia about the cm, I^{cm}_{ab}.

- Let the CM be shifted by $\vec{d}$ from the O, i.e. $\vec{r}_{cm} = \vec{r}_O + \vec{d}$.

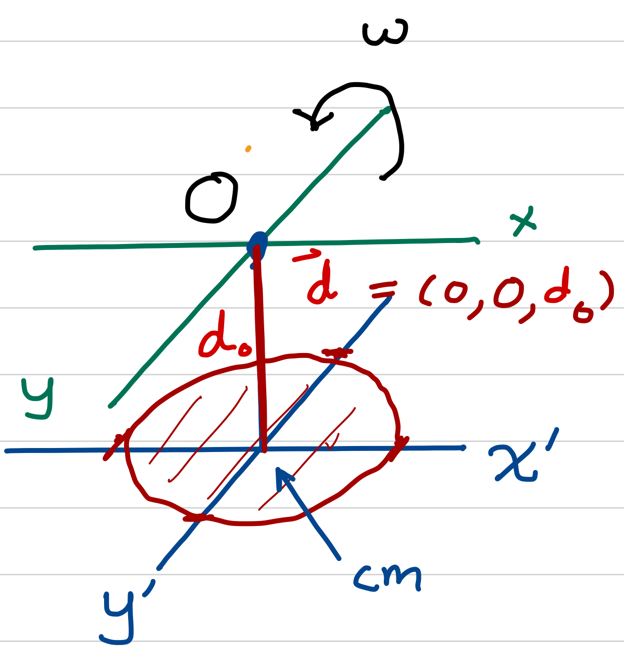
Then the moment of inertia about O is

$$I^O_{ab} = I^{cm}_{ab} + M\,(d^2\delta_{ab} - d_a d_b)$$

Shift matrix

See Tong notes for proof

Example:



- Suppose you wanted to know the kinetic energy if the object (the thing in red) was spinning around the y axis, with ω.

You can use the // axis theorem to determine I^O and then evaluate the KE.

$$I^O_{ab} = \underbrace{\begin{pmatrix} \tfrac{1}{4}MR^2 & 0 & 0 \\ 0 & \tfrac{1}{4}MR^2 & 0 \\ 0 & 0 & \tfrac{1}{2}MR^2 \end{pmatrix}}_{I_{cm}} + M \begin{pmatrix} d_0^2 & & \\ & d_0^2 & \\ & & d_0^2 - d_0^2 \end{pmatrix}$$

shift

- So the kinetic energy is for rotations around y

$$T = \frac{1}{2}\omega I^{o}_{yy}\omega$$

$$= \underbrace{\frac{1}{2}\left(\frac{1}{4}mR^2\right)\omega^2}_{\text{angular KE about CM}} + \underbrace{\frac{1}{2}Md_o^2\omega^2}_{\text{KE of CM motion } v_{cm}=d_o\omega}$$

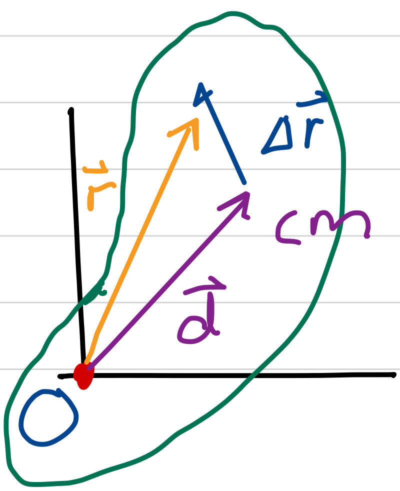
$$= \frac{1}{2}I^{cm}_{yy}\omega^2 + \frac{1}{2}Mv_{cm}^2$$

- In general for an object rotating around a fixed O with O displaced by $\vec{d}$ from the CM find:

$$T = \frac{1}{2}\omega_a I^{cm}_{ab}\omega_b + \frac{1}{2}Mv_{cm}^2 = \frac{1}{2}\omega_a I^{o}_{ab}\omega_b$$

where $\vec{v}_{cm} = \vec{\omega}\times\vec{d}$, and $\vec{d}$ is the position of the center of mass, relative to O. The parallel axis theorem lets the two expressions for KE agree.

Picture:



So: $\vec{r} = \vec{d} + \Delta\vec{r}$,

thus, $\vec{\omega}\times\vec{r} = \vec{\omega}\times\vec{d} + \vec{\omega}\times\Delta\vec{r}$

$$\vec{V} = \vec{V}_{cm} + \Delta\vec{V}.$$

Similarly

$$(\vec{L})_a = I^{cm}_{ab}\,\omega_b + (\vec{r}_{cm} \times M\vec{v}_{cm})_a = I^{O}_{ab}\,\omega_b$$

with $\vec{v}_{cm} = \vec{\omega} \times \vec{d}$