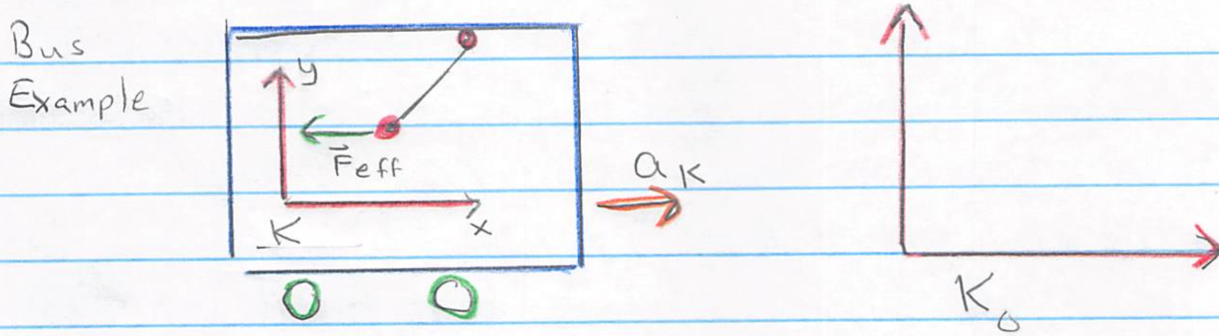
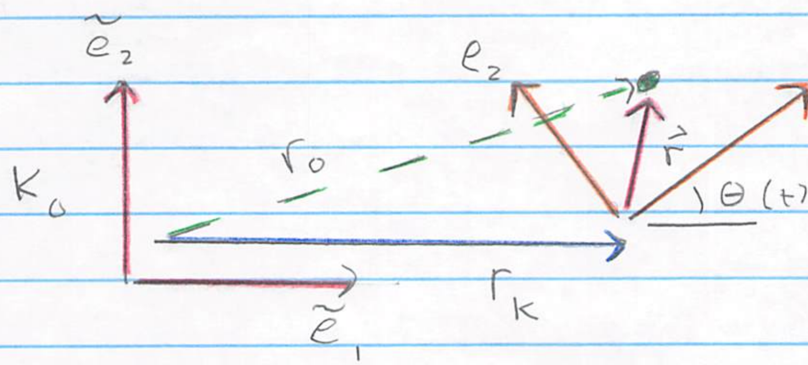
Motion in Accelerating Frames

Bus Example



- The Lab frame (coordinates) is denoted by K_0
- The observers frame is K, and is accelerating with $\vec{a}_k$ in inertial frame K_0. This gives rise to additional effective forces $\vec{F}_{eff}$ in K



- Now consider the general case where the frame is rotating, as well as translating:

So

$$\vec{r}_0 = \vec{r} + \vec{r}_K$$

← position of origin of non-inertial frame K

$$\frac{d\vec{r}_0}{dt} = \frac{d\vec{r}}{dt} + \frac{d\vec{r}_K}{dt}$$

$$V_0 = \left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega}\times\vec{r} + \vec{V}_K$$

Here

- $\left(\frac{d\vec{r}}{dt}\right)_r = \vec{V}_r$ is the time derivative in the moving frame assuming the basis vectors $\{\vec{e}_a\}$ are constant

- $\vec{\omega} \times \vec{r}$ Reflects the fact that the basis vectors are rotating in time

- So concluding and extending by taking another $\frac{d}{dt}$:

$$\vec{V}_0 = \vec{V}_r + \vec{\omega} \times \vec{r} + \vec{V}_K$$

$$\vec{a}_0 = \frac{d\vec{V}_0}{dt} = \left(\frac{d\vec{V}_r}{dt} + \vec{\omega} \times \vec{V}_r\right) + \vec{\omega} \times \left(\left(\frac{d\vec{r}}{dt}\right)_r + \vec{\omega} \times \vec{r}\right)$$

$$+ \Big(\underbrace{\left(\frac{d\omega}{dt}\right)_r}_{\dot{\vec{\omega}}_r} + \underbrace{\vec{\omega} \times \vec{\omega}}_{=0}\Big) \times \vec{r} + \vec{a}_K$$

$$\boxed{\vec{a}_0 \equiv \vec{a}_r + 2\vec{\omega} \times \vec{V}_r + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}}_r \times \vec{r} + \vec{a}_K}$$

- The forces obey

$$\vec{F} = m\vec{a}_0$$

Or defining $\vec{F}_{eff}$, an effective force:

$$\boxed{\vec{F}_{eff} = \vec{F} \underbrace{- 2m\,\vec{\omega}\times\vec{v}_r}_{1} \underbrace{- m\,\vec{\omega}\times(\vec{\omega}\times\vec{r})}_{2} \underbrace{- m\,\dot{\vec{\omega}}_r\times\vec{r}}_{3} \underbrace{- m\vec{a}_k}_{4}}$$

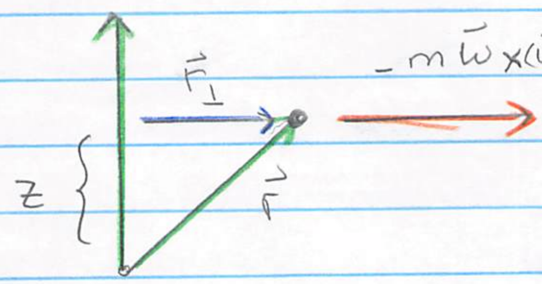
we have

$$\boxed{\vec{F}_{eff} = m\vec{a}_r}$$

The Terms are:

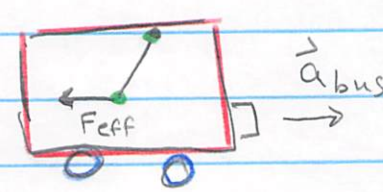
1. The coriolis term (see below)

2. The centrifugal term



$$-m\vec{\omega}\times(\vec{\omega}\times\vec{r}) = +m\omega^2\,\vec{r}_\perp$$

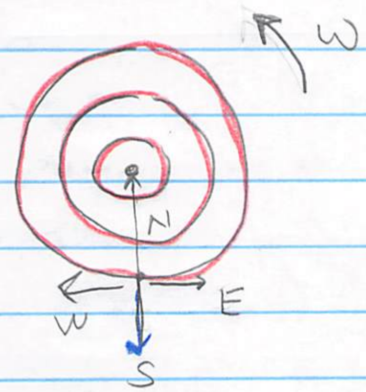
3. $\dot{\vec{\omega}}_r\times\vec{r}$ rather exotic



4. $-m\vec{a}_k$ the "bus" term

Example of Coriolis Force

- Consider a rocket fired south from the north pole with velocity $\vec{v}$



$$\vec{\omega} = \omega \hat{z}$$

$$\omega = \frac{2\pi \text{ rad}}{\text{day}}$$

$$\omega^2 R_e = 3.38 \text{ cm}^2/\text{sec}$$

- After setting up coordinate with x = east, y = north and z = up

$\simeq 0.03\, g$

$$m \frac{d^2\vec{r}}{dt^2} = -2m(\vec{\omega} \times \vec{v})$$

$$= -2m(\omega \hat{z} \times (-\hat{y})v)$$

$$m \frac{d^2\vec{r}}{dt^2} = -2m\omega v \hat{x}$$

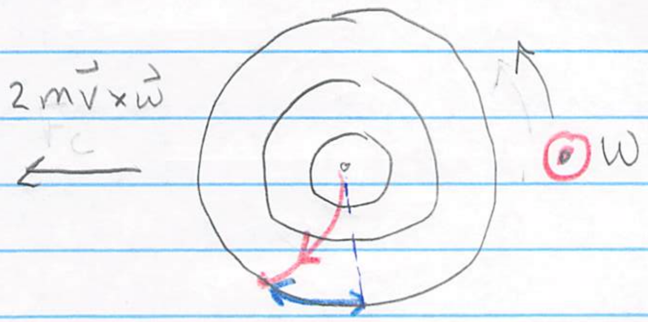
constant force

$$x(t) = -\omega v t^2 \hat{x}$$

- The minus sign indicates a Westward deflection

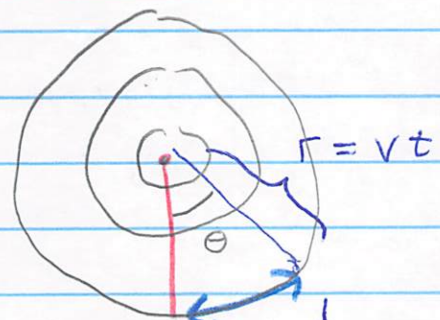
Rotating Frame picture: Coriolis Force

$F_c = 2m\vec{v}\times\vec{\omega}$



$d = \omega v t^2$

Inertial Frame Picture: No Force



$d' = r\theta = (vt)(\omega t) = v\omega t^2$

Of course the deflection is the same

Non Inertial Frames and the Lagrangian

- The treatment of non-inertial frames is also easily handled in the lagrangian framework. I will treat rotations only. Landau treats the general case

$$L = \frac{1}{2} m \vec{v}_0^{\,2} - U$$

- Now $\vec{v}_0 = (\vec{v}_r + \vec{\omega} \times \vec{r})$

- So the lagrangian for the moving frame coordinates is

$$L = \frac{1}{2} m \vec{v}_r^{\,2} + \underbrace{m \vec{v}_r \cdot \vec{\omega} \times \vec{r}}_{\text{coriolis}} + \underbrace{\frac{1}{2} m (\vec{\omega} \times \vec{r})^2}_{\text{Centrifugal potential}} - U$$

- Then the canonical momentum is:

$$\vec{p}_a^{\,can} = \frac{\partial L}{\partial v_r^a} = m v^r_a + m(\vec{\omega} \times \vec{r})_a = m(\vec{v}_0)_a$$

$\uparrow$ kinetic momentum

- We will assume $\vec{\omega}$ constant. Then there is a "first integral" or hamiltonian function:

$$h = \frac{\partial L}{\partial \vec{v}_r} \cdot \vec{v}_r - L = \frac{1}{2} m \vec{v}_r^{\,2} - \frac{1}{2} m (\vec{\omega} \times \vec{r})^2 - U(r)$$

$\nwarrow$ centrifugal potential but no coriolis term

which is constant in time.

- We are recalling that for a Lagrangian:

$$L = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + b_i \dot{q}^i - U(q)$$

Then

$$p_i = a_{ij} \dot{q}^j + b_i$$

And

$$h = p_i \dot{q}^i - L = \frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + U(q)$$

- i.e. the b's cancel. Here we see that this means that the coriolis term does not contribute to h

$$\boxed{h = \frac{1}{2} m v_r^2 + \left[U - \frac{1}{2} m (\vec{\omega} \times \vec{r})^2 \right]}$$

- Note that h is not $E_0 = \frac{1}{2} m v_0^2 + U(r)$, but i it is constant in time.

Using the fact that

$$\vec{v} = V_0 - \vec{\omega} \times \vec{r}$$

We find

$$h = \underbrace{\left(\frac{1}{2} m v_0^2 + U \right)}_{E_0} - m \vec{v}_0 \cdot (\vec{\omega} \times \vec{r}_0)$$

Re-arranging the cross product:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$$

We have with $\vec{p}_0 = m\vec{v}_0$

$$h = E_0 - \vec{\omega} \cdot (\vec{r}_0 \times \vec{p}_0)$$

i.e

$$\boxed{h = E_0 - \vec{\omega} \cdot \vec{L}}$$

- This formula shows how the energy changes when we go to a rotating frame. Although we derived it for a single particle, it is quite general holding for a system of classical or quantum mechanical interacting particles