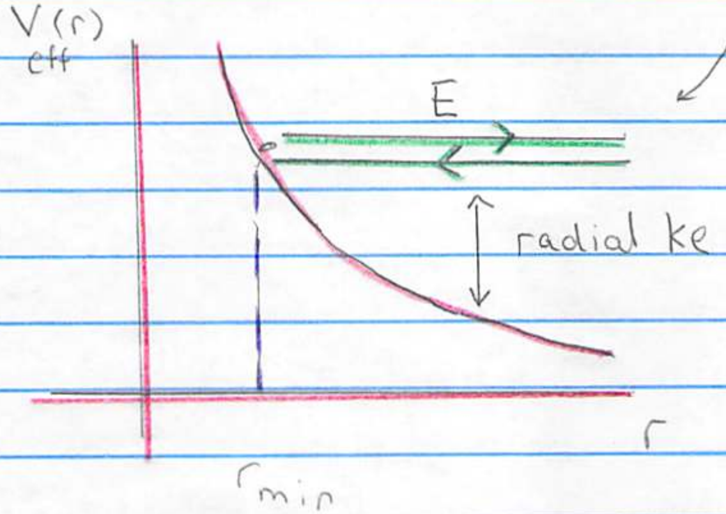
Scattering

- For definiteness take:

$$U(r) = +\frac{k}{r} \quad \text{and} \quad \mu = m_1 = m \quad m_2 \to \infty$$

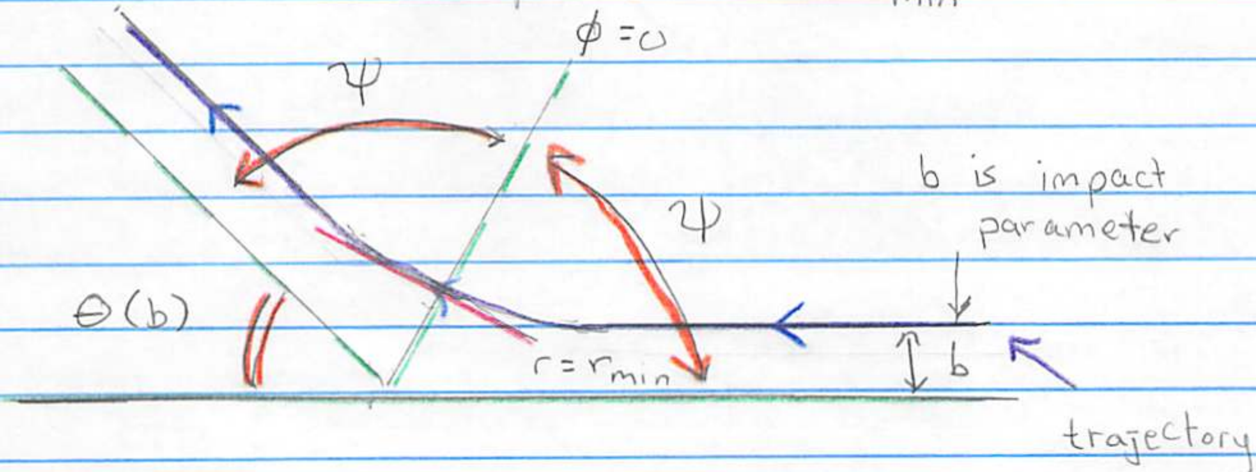
- Then $V_{eff} = \dfrac{\ell^2}{2mr^2} + U(r)$



this describes a particle flying in and out

- The particle has angular mom $= \ell$. So, the angle ϕ changes by Ψ as we go from $r = r_{min}$ to $r \to \infty$

- Then in coordinate space:



$$\Theta(b) = \pi - 2\Psi \equiv \text{scattering angle}$$

- The angular momentum is related to b:

$$\ell = m v b = \sqrt{2mE}\, b$$

- Now the angular change, ψ, on the way out is:

$$\psi = \frac{\ell}{\sqrt{2m}} \int_{r_{min}}^{\infty} \frac{dr/r^2}{(E - V_{eff}(r))^{1/2}}$$

- Introducing the same dimensionless parameters:

opposite sign from ellipse

$$r_0 \equiv \frac{\ell^2}{mk} \qquad \varepsilon \equiv \frac{E}{\ell^2/2mr_0^2} \qquad V = \frac{V_{eff}}{\varepsilon_0} = \left(\frac{1}{\underline{r}^2} + \frac{2}{\underline{r}}\right)$$

$$u \equiv r_0/r \qquad \varepsilon = 2E\ell^2/mk^2 \qquad V = (u+1)^2 - 1$$

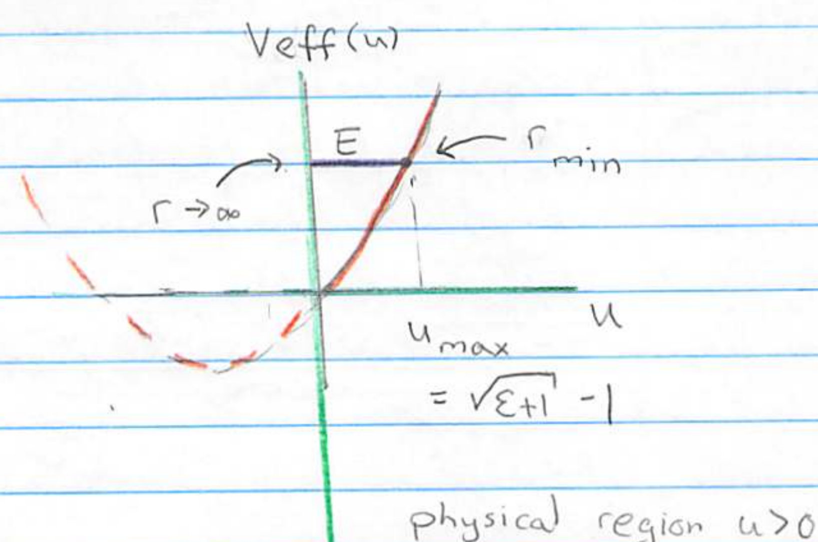
opposite sign from ellipse

- Then

$$\psi = \int_0^{u_{max}} \frac{du}{(\varepsilon + 1 - (u+1)^2)^{1/2}}$$

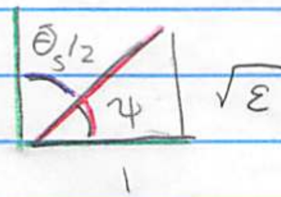
$$= \operatorname{atan}(\sqrt{\varepsilon})$$

function of E



- So restoring units, using $\theta_s/2 = \frac{\pi}{2} - \psi$:

$$\cot \theta_s/2 = \sqrt{\frac{2E\ell^2}{mk^2}} = \sqrt{\mathcal{E}}$$



$\tan \psi = \sqrt{\mathcal{E}}$

$$\boxed{\cot \theta_s/2 = \sqrt{\left(\frac{2Eb}{k}\right)^2} = \frac{2Eb}{k}}$$

- So now we have expressed the scattering angle θ_s in terms of the integrals of the motion:

$$E \quad \text{and} \quad b = \frac{\ell}{\sqrt{2mE}}$$

And our job as classical mechanists is done.

- But the result is often expressed differently in terms of <u>cross-section</u>

Cross Section

- We consider a beam with Luminosity $\mathcal{L}$ = "intensity" ≡ number of particles per area per second, incident upon a target

(See picture on next page)

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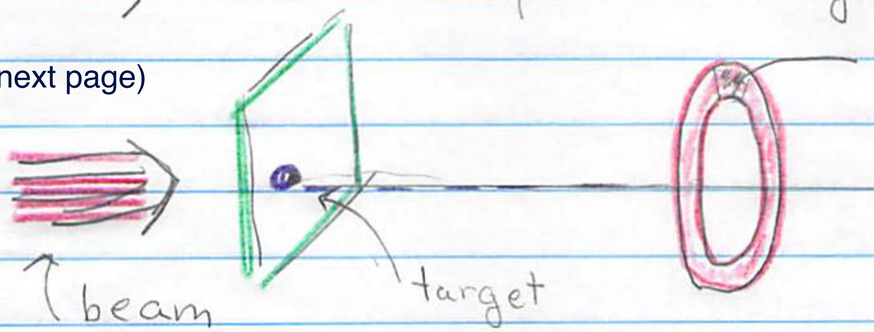


- There is an event rate, dR, which particles scatter into the solid angle $d\Omega$. By definition the cross section $d\sigma$ is:

$$dR = \mathcal{L} \, d\sigma$$

"the event rate is the cross section × the Luminosity"

- Of course this rate/$d\sigma$ is proportional to the (detector) dector size:

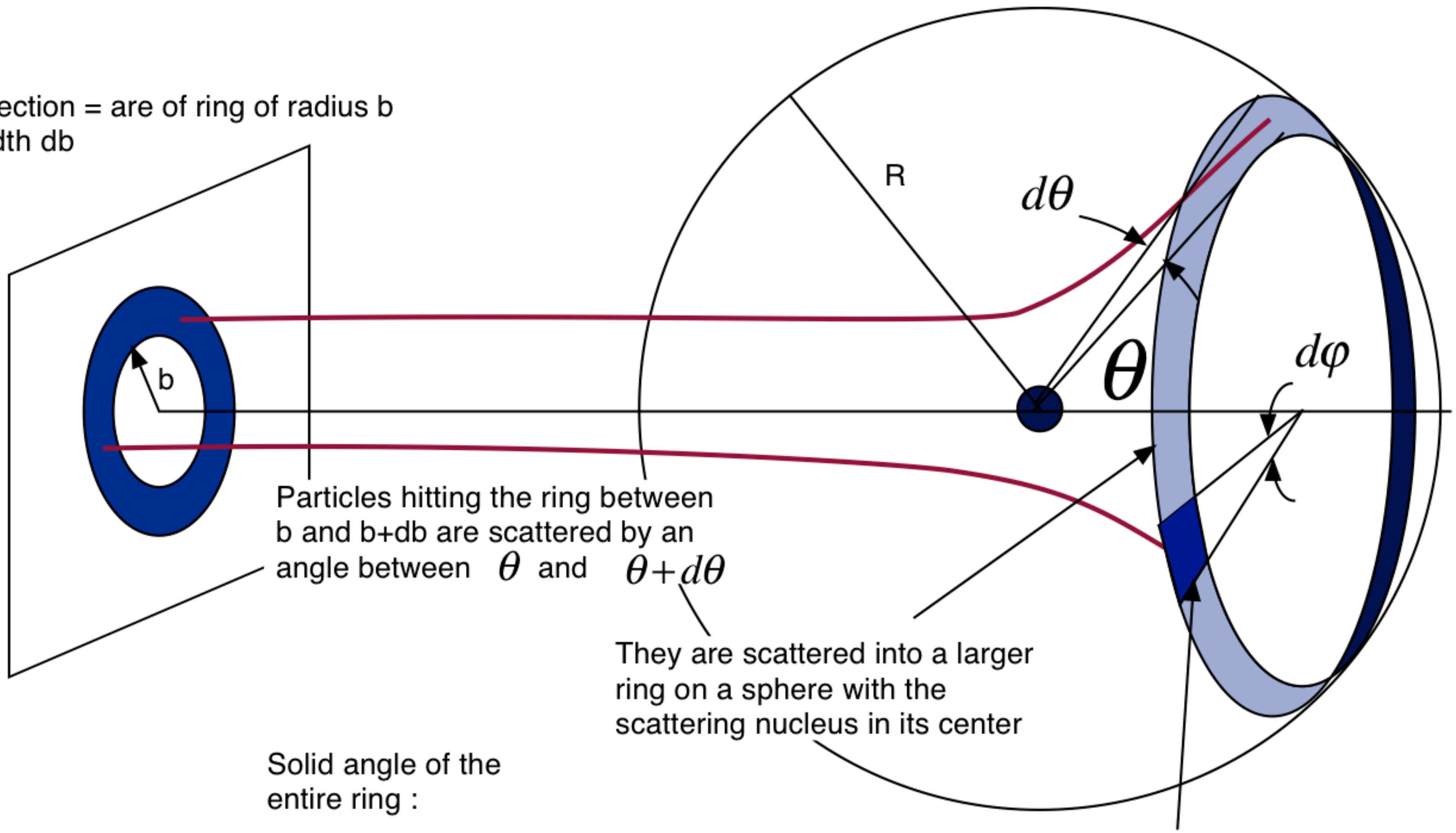
$$\bigstar \quad d\sigma \equiv \frac{d\sigma}{d\Omega} d\Omega = \frac{d\sigma(\theta)}{d\Omega} \overbrace{2\pi \sin\theta \, d\theta}^{d\Omega}$$

← differential cross section: finite object of study

- Now the rate for particles to scatter with impact parameter between b and $b+db$ is

$$dR_b = \underbrace{\mathcal{L}}_{\#/(\text{Area}\cdot\text{sec})} \; \underbrace{2\pi b \, db}_{\text{area of ring}}$$

Figure credit: Werner Boeglin, Florida International University



Solid angle of the
entire ring :

$$d\Omega = \frac{2\pi R \sin(\theta) R d\theta}{R^2} = 2\pi \sin(\theta) d\theta$$

solid angle of
small area:

$$d\Omega = \frac{d\varphi R \sin(\theta) R d\theta}{R^2} = \sin(\theta) d\theta d\varphi$$

- Then for each b there is a scattering angle $\theta(b)$. So the rate with $d\theta = d\Omega / 2\pi \sin\theta$

$$dR = \mathcal{L}\, 2\pi b \left|\frac{db}{d\theta}\right| d\theta$$

(the scattering rate to scatter between theta and theta + dtheta)

$$dR = \mathcal{L}\, \frac{b}{\sin\theta} \left|\frac{db}{d\theta}\right| 2\pi \sin\theta\, d\theta \equiv \mathcal{L}\, d\sigma$$

So comparison with ☆ (two pages back)

$$\boxed{\frac{d\sigma}{d\Omega}(\theta) = \frac{b}{\sin\theta}\left|\frac{db}{d\theta}\right|}$$

(This is a general formula that you could possibly need on homework/comps)

- Now for the case of coulomb scattering c $\cot\theta/2 = 2Eb/k$. So substituting and differentiating

$$\boxed{\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}}$$

used $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$

$b = \frac{k}{2E}\cot(\theta/2)$

Rutherford cross-setion