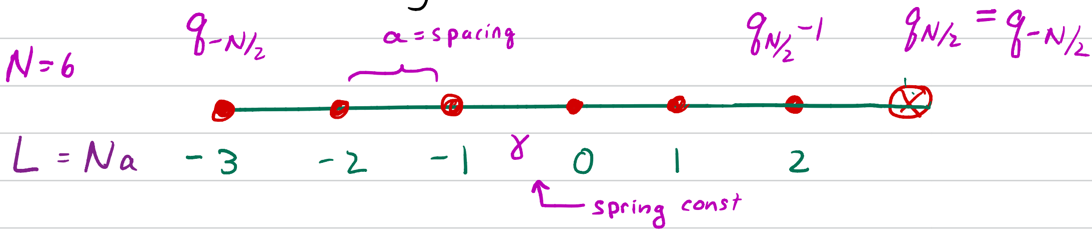
Wave propagation

- Lets return to the discrete case and discuss boundary conditions



- Consider a finite chain with periodic B.C. q_j with $j = -N/2, \ldots, N/2 - 1$

$$q_{N/2} = q_{-N/2}$$

- Then exactly as before we would look for eigen solutions. (We will not use the summation convention in this section)

$$q_j = A_k E^k_j e^{i\omega t} \qquad \text{(no sum)}$$

- Then as before we find eigen-vectors & values

$$E^k_j = e^{ikx_j} \qquad \omega = \pm\, \omega(k)$$

Now however we must have $q_{N/2} = q_{-N/2}$ or

$$k N a = 2\pi m \implies k_m = \frac{2\pi}{L} m$$

integer

- However not all m lead to independent solution since ($L = Na$, $x_j = ja$, $k_m = 2\pi m / L$)

$$e^{i k_m x_j} = e^{i 2\pi j m / N}$$

Thus, $m + N$ yields the same solution as m, and we can take the set

$$k_m = \frac{2\pi}{L} m \qquad \underbrace{m = -\frac{N}{2} \ldots\ldots \frac{N}{2} - 1}_{N \text{ modes}}$$

- We have found N eigen modes and we can use this as a basis for the complete solution :

$$q_j = \frac{1}{L} \sum_m A_m e^{i k_m x_j - i\omega(k_m) t} + B_m e^{i k_m x_j + i\omega(k_m) t}$$

inserted for convenienc

If q_j is real the term with $-k_m$ in the B terms, must match the term with k_m in the A - terms leading to $B_{-m} = A_m^*$, ie.

$$\boxed{q_j = \frac{1}{L} \sum_m A_m e^{i k_m x_j - i\omega(k) t} + c.c.}$$

complex conjugate of first term

- Finally Lets take the limit $N \rightarrow \infty$

$$\sum_m \longrightarrow \int dm = \int \frac{L\,dk}{2\pi}$$

$$A_m \longrightarrow A(k)$$

Leading to a fourier integral representation :

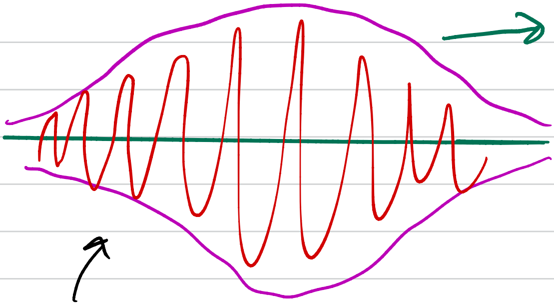
$$\boxed{q(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx - i\omega_k t} + c.c.}$$

- Lets write out c.c.

$$\boxed{q(t,x) = Re\left[\int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx - i\omega(k) t} \right]}$$

Propagation of Wave Packets

- Now let us consider the propagation of a superposition of waves



typical form

- The wave solution takes the form

$$q(x,t) = \int \frac{dk}{2\pi} A(k)\, e^{ikx - i\omega(k)t}$$

we will take complex waves here. In general one would insert "real part" everywhere but that would lead to unnecessary complications, with no insight.

- The wave form at $t=0$ determines the coefficients $A(k)$

$$q(x,0) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} A(k)\, e^{ikx} \Longrightarrow A(k) = \int_{-\infty}^{\infty} dx\, u(x,0)\, e^{-ik\cdot x}$$

- These are fourier transforms, lets recall some properties of Fourier transforms. (A tutorial type problem is optional on the next homework)

$$\hat{f}(k) \equiv \int_{-\infty}^{\infty} dx\, e^{-ikx} f(x) \qquad \text{Definition}$$

$$f(x) \equiv \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, e^{ikx} \hat{f}(k) \qquad \text{Inverse}$$

- Then DO THE HOMEWORK

Space		k-Space
gaussian $\sigma \to 1/\sigma$ $\quad G(x) = e^{-x^2/2\sigma^2}$	$\Longleftrightarrow$	$\hat{G}(k) = \sqrt{2\pi\sigma^2}\, e^{-k^2\sigma^2/2}$
phase makes translation $\quad f(x) = e^{ik_0 x} g(x)$	$\Longleftrightarrow$	$\hat{f}(k) = \hat{g}(k-k_0)$
$f(x) = g(x-x_0)$	$\Longleftrightarrow$	$\hat{f}(k) = e^{-ikx_0} \hat{g}(k)$

- The uncertainty principle states that if

$$(\Delta x)^2 \equiv \int dx\, |f(x)|^2 (x-\bar{x})^2$$

$$(\Delta k)^2 \equiv \int \frac{dk}{2\pi} |\hat{f}(k)|^2 (k-\bar{k})^2, \quad \text{find}$$

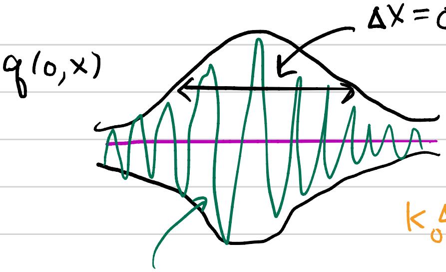
$\Delta k\, \Delta x \geq 1/2$, with equality holding uniquely for Gaussian.

- The typical wave packet

$$q(x,0) = G(x)\, e^{ik_0 x} \qquad G(x) \equiv e^{-x^2/2\sigma^2}$$

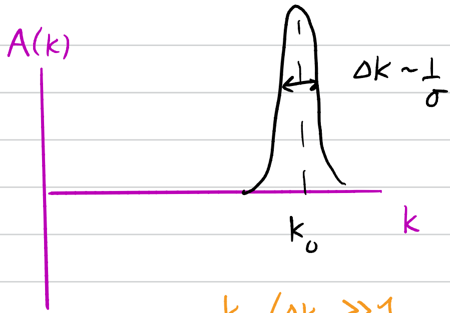
Gaussian

- Then use table



$k_0 \Delta x \gg 1$

Wavelength determines k_0



$k_0 / \Delta k \gg 1$

- We have $A(k) = \sqrt{2\pi\sigma^2}\; e^{-\frac{1}{2}(k-k_0)^2\sigma^2}$

- No we have at future times

$$q(x,t) = \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, A(k)\, e^{ikx - i\omega(k)t}$$

Now we can expand near k_0, since $A(k)$ is $A(k)$ is narrow

$$\omega(k) \simeq \omega_0 + U(k-k_0)$$

$\omega(k_0)$ $\qquad U = \left.\frac{d\omega}{dk}\right|_{k_0} \equiv$ group velocity

So:

So

$$q(x,t) = \underbrace{e^{i(\mathcal{U} k_0 - \omega_0)t}}_{\text{phase}} \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(x - \mathcal{U} t)} A(k)$$

- Thus since $A(k)$ was the FTrans of $q(x,0)$

$$q(x,t) = e^{i(\mathcal{U}k_0 - \omega_0)t} q(x - \mathcal{U}t, 0)$$

Since $q(x,0) = G(x) e^{ik_0 x}$ we have

$$q(x,t) = e^{i(kx_0 - \omega_0 t)} G(x - \mathcal{U}t)$$

- So we see that the envelope $G(x)$ of the wave form is shifted by $\mathcal{U}t$ with

$$\boxed{\mathcal{U} = \left.\frac{d\omega}{dk}\right|_{k_0}}$$

← this is the speed of the packet

group velocity

- Notice that the trajectory follows the points of stationary phase:

$$\frac{\partial}{\partial k}\left(kx - \omega(k)t\right) = 0 \implies x - \mathcal{U}t = 0$$

The variational problem and dispersion

- Lets recall for an infinite chain of springs, found the dispersion curve:

$$\omega(k) = \pm 2\omega_0 \sin\left(\frac{ka}{2}\right) \qquad \omega_0 = \sqrt{\frac{\gamma}{m}}$$

with $v_0 = \omega_0 a$

$$\simeq \pm v_0 k \left(1 - \frac{(ka)^2}{24} + \ldots\ldots\right)$$

- Let's see how the continuous theory can reproduce this result. Consider the action

$$S = \int dt\, dx \left(\frac{1}{2}\mu (\partial_t q)^2 - \frac{1}{2} Y (\partial_x q)^2 + C(\partial_x^2 q)^2 \right)$$

Then we vary the action, e.g.

$$(\partial_x^2 q)^2 \longrightarrow (\partial_x^2 q + \partial_x^2 \delta q)^2$$

$$\simeq (\partial_x^2 q)^2 + 2(\partial_x^2 q)\, \partial_x^2 \delta q$$

- So

$$\delta S = \int dt\, dx \left(\mu\, \partial_t q\, \partial_t \delta q - Y \partial_x q\, \partial_x \delta q + 2C(\partial_x^2 q)\, \partial_x^2 \delta q \right)$$

2x integrate by parts

integrate by parts ignoring all boundary terms

- Find

$$\delta S = \int dt\,dx \left[-\partial_t(\mu\,\partial_t q) + \partial_x(Y\,\partial_x q) + 2\,\partial_x^2\big(C(\partial_x^2 q)\big) \right] \delta q$$

So the EOM is

$$-\partial_t(\mu\,\partial_t q) + \partial_x(Y\,\partial_x q) + 2\,\partial_x^2\left[C(\partial_x q)^2\right] = 0$$

- Treating the parameters as constants we substitute $q = A e^{ikx - i\omega t}$

$$\mu\,\omega^2 - Y k^2 + 2 C k^4 = 0$$

For $C k^4 \ll Y k^2$ we find $\omega = \pm \left(Y/\mu\, k^2 - \frac{C}{\mu} k^4 \right)^{1/2}$

$$\omega \simeq \pm \sqrt{\frac{Y}{\mu}}\, k \left(1 - \frac{C}{Y} k^2 \ldots\ldots \right)$$

- Thus by choosing C and Y

$$\sqrt{\frac{Y}{\mu}} = \omega_0 a \quad \text{and} \quad \frac{C}{Y} = \frac{a^2}{24}$$

we can reproduce the microscopic theory.

☆ In general by adding more and more derivatives to the action, such as $(\partial_x^3 q)^2$, even higher terms of the micro-theory can be reproduced by tuning the "low-energy" constants such as C in this case.