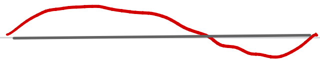
Energy Momentum in the Waves

- The canonical momentum density for the Lagrangian:

$$L = \int dx \overbrace{\tfrac{1}{2}\mu(\partial_t y)^2 - \tfrac{1}{2}T(\partial_x y)^2}^{\text{Lagrange density } \mathcal{L}}$$

$y(t,x)$



take a string for example

is

$$\pi^0 = \frac{\partial \mathcal{L}}{\partial(\partial_t y)} = \mu\, \partial_t y$$

- Then the Hamiltonian function $h = \sum_i p_i \dot{q}^i - L$ for a discrete system becomes

$$h = \int dx\, \pi^0 \partial_t y - \mathcal{L}$$

or

$$h = \int dx \underbrace{\left(\tfrac{1}{2}\mu(\partial_t y)^2 + \tfrac{T}{2}(\partial_x y)^2\right)}_{\text{energy density } \varepsilon}$$

- We expect the total energy h to be conserved. We also expect that the energy is conserved over any patch, except for the energy that flows out of the patch

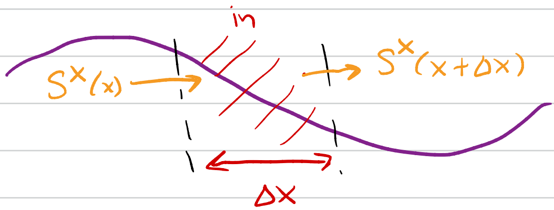
- The most (only?) way to achieve local energy conservation is to have the energy satisfy a conservation law

energy flux S^x: energy / time

$$\boxed{\partial_t \mathcal{E} + \partial_x S^x = 0}$$

energy density
energy / length

(In 3-dimensions $\mathcal{E}$ = energy / vol , S^x = energy / Area / time)



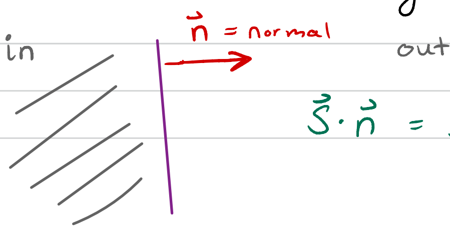
- Then integrating over Δx we have

$$\partial_t \left[\int_{\Delta x} dx \, \mathcal{E} \right] = - \int_{\Delta x} \partial_x S^x$$

$$= - \left(S^x(x+\Delta x) - S^x(x) \right)$$

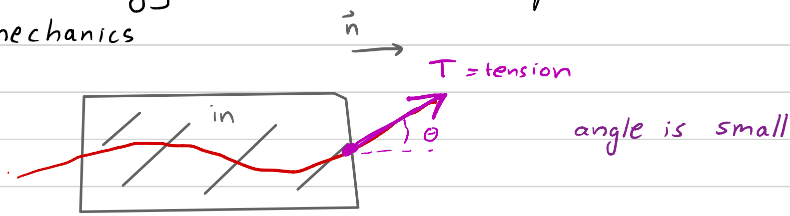
energy in Δx bndry terms, nothing inside can change the energy

Here S^x is the work by the left side of the interface (the inside) on the right (the outside):



$$\vec{S} \cdot \vec{n} = S^x$$

- The energy flux can be computed with simple mechanics



$$\frac{dW}{dt} = T\sin\theta \, \partial_t y = T \frac{\partial y}{\partial x} \frac{\partial y}{\partial t}$$

Here dW/dt is the work done by the tension (outside) on the string inside the box. We want S^x the work per time by the inside (left side) on the outside so we multiply by -1.

$$\boxed{S^x = -T\left(\frac{\partial y}{\partial x}\right)\left(\frac{\partial y}{\partial t}\right)}$$

We will derive this conservation law

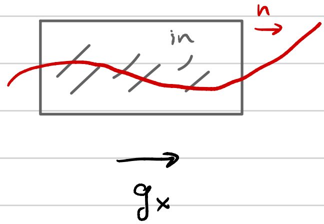
$$\partial_t \mathcal{E} + \partial_x S^x = 0$$

and a similar one for momentum in the next section. The conservation law for momentum reads

$$\boxed{\partial_t g_x + \partial_x T^x{}_x = 0}$$

(1) Here g_x is the momentum in the x-direction per length. (See below)

(2) $T^x{}_x$ is the force by the left hand side (inside) on the outside. $-T^x{}_x$ is force by out on in.



$$\partial_t g_x = \underbrace{-\partial_x T^x{}_x}_{\text{net force/volume}}$$

A similar (Newton Law) derivation gives

$$g_x = -\mu \frac{\partial y}{\partial x}\frac{\partial y}{\partial t}$$

$$T^x{}_x = \frac{T}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2$$

We will derive these expressions and the conservation Laws from the EOM next.

The canonical stress tensor

- The canonical stress tensor summarizes these conservation laws. First define

$$x^\mu \equiv (x^0, x^1) \equiv (t, x)$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu y)}$$

Then for Lagrange density $\mathcal{L}(y, \partial_\mu y, x)$

$$\boxed{T^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu y)} \partial_\nu y + \mathcal{L}\,\delta^\mu_\nu}$$

To rember this think $-p\dot{q} + L = -H$ but p gets replaced by $\pi^\mu = \partial \mathcal{L} / \partial(\partial_\mu y)$.

- We will show now

$$\boxed{\partial_\mu T^\mu{}_\nu = 0}$$

two equations one for $\nu=0$ and one for $\nu=1$

- For the lagrange density $\mathcal{L} = \frac{1}{2}\mu(\partial_t y)^2 - \frac{T}{2}(\partial_x y)^2$

$$T^0{}_0 = -\varepsilon$$

$$= -\frac{\partial \mathcal{L}}{\partial(\partial_t y)} \partial_t y + \mathcal{L}\,\underbrace{\delta^0_0}_{=1} = -\varepsilon$$

And

$$T^x{}_0 = -\frac{\partial \mathcal{L}}{\partial(\partial_x y)} \partial_t y + \mathcal{L}\,\delta^x{}_0$$

$$= + T \partial_x y \, \partial_t y = - S^x$$

So

$\partial_t T^0{}_0 + \partial_x T^x{}_0 = 0$ encodes the energy conservation law $\boxed{\partial_t \varepsilon + \partial_x S^x = 0}$

- Similarly:

$$T^0{}_x = -\frac{\partial \mathcal{L}}{\partial(\partial_t y)} \partial_x y = -\mu \, \partial_t y \, \partial_x y \equiv g_x$$

$$T^x{}_x \equiv -\frac{\partial \mathcal{L}}{\partial(\partial_x y)} \partial_x y + \mathcal{L}\,\delta^x{}_x = T \partial_x y \, \partial_x y + \mathcal{L}$$

$$= \frac{T}{2}(\partial_x y)^2 + \frac{\mu}{2}(\partial_t y)^2$$

And the momentum conservation law

$$\boxed{\partial_t g_x + \partial_x T^x{}_x = 0}$$

Proof that $\partial_\mu T^\mu{}_\nu = 0$

$$\mathcal{L}(y, \partial_\mu y, x^\mu)$$

- The proof parallels the proof that $\partial_t h = 0$. Just differentiate and use the EOM.

$$\partial_\mu T^\mu{}_\nu = \partial_\mu\left(-\frac{\partial\mathcal{L}}{\partial(\partial_\mu y)}\partial_\nu y + \mathcal{L}\,\delta^\mu{}_\nu\right)$$

$$= \partial_\mu\left(-\frac{\partial\mathcal{L}}{\partial_\mu y}\right)\partial_\nu y - \frac{\partial\mathcal{L}}{\partial(\partial_\mu y)}\partial_\mu\partial_\nu y$$

cancel

$$+ \frac{\partial\mathcal{L}}{\partial y}\partial_\nu y + \frac{\partial\mathcal{L}}{\partial(\partial_\mu y)}\partial_\nu(\partial_\mu y) + \partial_\nu\mathcal{L}$$

We see that ___ $= 0$, by the EOM

$$-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial_\mu y}\right) + \frac{\partial\mathcal{L}}{\partial y} = 0$$

leading to the result

$$\partial_\mu T^\mu{}_\nu = \frac{\partial\mathcal{L}}{\partial x^\nu}.$$

When $\mathcal{L}$ does not depend explicitly on space-time, we find:

$$\boxed{\partial_\mu T^\mu{}_\nu = 0}$$