Integration of the EOM for effectively 1D systems

Landau #11

- For lagrangians of the form

$$L = \frac{1}{2} \hat{m}(q)\, \dot{q}^2 - V_{eff}(q)$$

we can always integrate the EOM since

$$E = \frac{1}{2} \hat{m}(q)\, \dot{q}^2 + V_{eff}(q) = \text{constant}$$

- Solving for $\dot{q}$

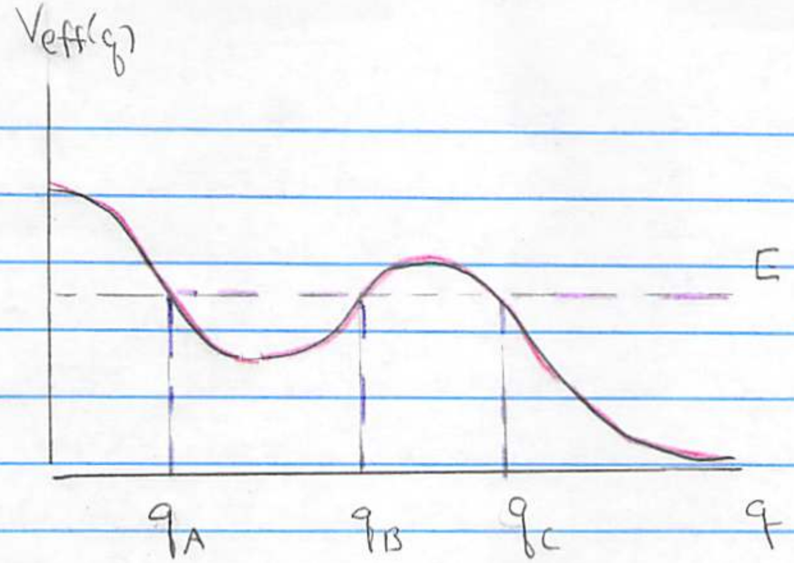
$$\frac{\hat{m}}{2} \left(\frac{dq}{dt}\right)^2 = E - V_{eff}(q)$$

Or

$$I(q) = \boxed{\int_{q_0}^{q} dq \sqrt{\frac{\hat{m}(q)}{2}} \frac{1}{(E - V_{eff}(q))^{1/2}} = t - t_0}$$

- This in principle determines $q(t)$, Though it may be hard to invert $I(q) = t - t_0$ for $q(t)$

- To analyze it further we set $\hat{m} = \text{constant}$

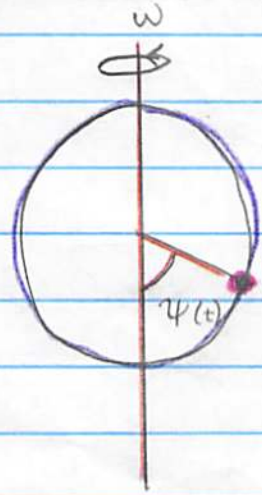


- For any given E the motion can take place only for $E > U(q)$
- For $q > q_C$ the motion is unbounded
- For $q_A < q < q_B$ the system will oscillate.

Starting at q_A, the time we reach q_B is half the period

$$\boxed{\frac{T}{2}(E) = \int_{q_A}^{q_B} dq \sqrt{\frac{\hat{m}}{2}} \frac{1}{(E - V_{eff}(q))^{1/2}}}$$

Example: Tong 2.5.1



- A bead can slide on a vertical circular rotating hoop. The hoop has angular velocity ω around the z-axis. Describe its motion

First
$$x(t) = a\sin\psi(t)\cos\omega t$$
$$y(t) = a\sin\psi(t)\sin\omega t$$
$$z(t) = -a\cos\psi(t)$$

- Then

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) = \frac{1}{2}ma^2(\dot{\psi}^2+\omega^2\sin^2\psi)$$

$$V = mgz = -mga\cos\psi(t)$$

So $L = T - V$

$$L = \frac{1}{2}\underbrace{ma^2}_{\hat{m}}\dot{\psi}^2 - \underbrace{\left[-\left(\frac{1}{2}ma^2\omega^2\sin^2\psi + mga\cos\psi\right)\right]}_{V_{eff}(\psi)}$$

aka energy

- There is a "first-integral" for this lagrangian

$$\boxed{h = p\dot{q} - L(q,\dot{q})}$$

$$= \frac{1}{2} m a^2 \dot{\psi}^2 + V_{eff}(\psi)$$

$$h = \frac{1}{2} m a^2 \dot{\psi}^2 - \frac{1}{2} m a^2 \omega^2 \sin^2\psi - mg\cos\psi = E$$

- Note this is <u>not</u> $T+V$

$$T+V = \frac{1}{2} m a^2 \dot{\psi}^2 + \frac{1}{2} m a^2 \omega^2 \sin^2\psi - mga\cos\psi$$

- Now using dimensional reasoning

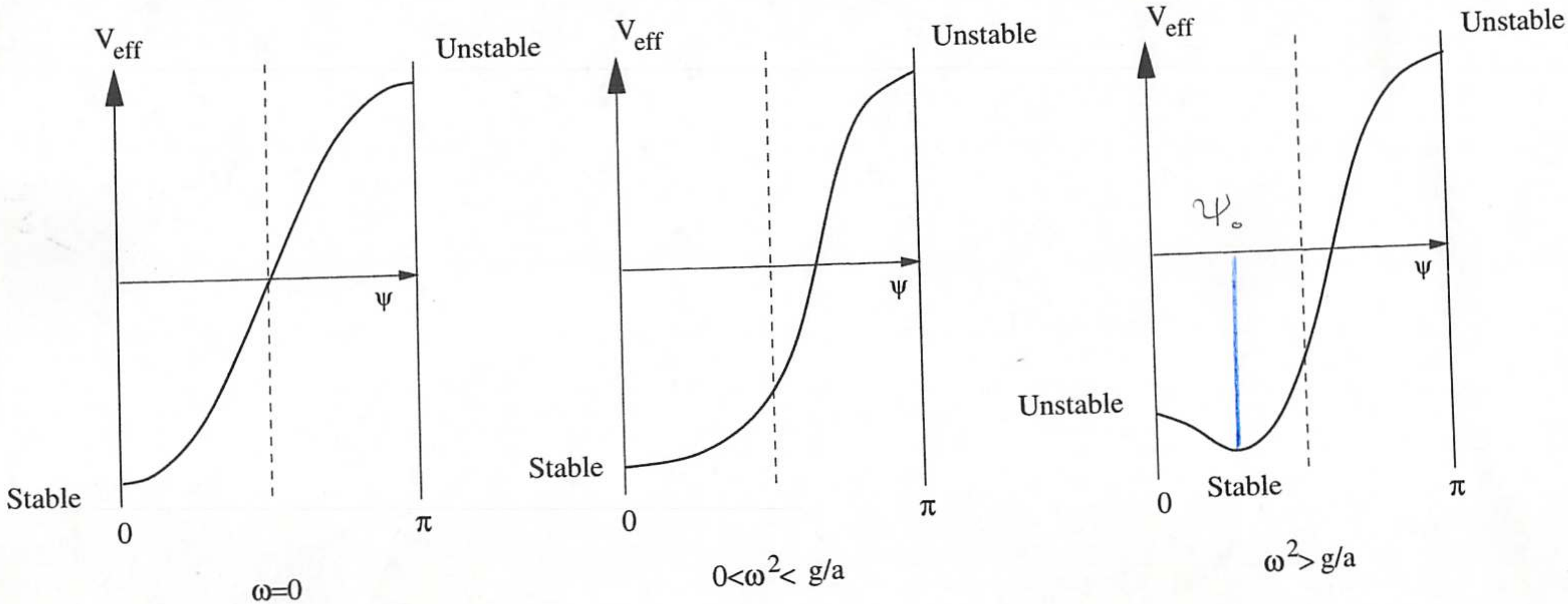
dimensionless v

$$\frac{V_{eff}}{mag} = -\left(\frac{a\omega^2}{g}\right)\frac{1}{2}\sin^2\psi - \cos\psi \equiv v(\psi)$$

ratio of centripetal acceleration to gravitational acceleration.

The next slide shows how the potential evolves as $a\omega^2/g$ is increased. For $a\omega^2/g > 1$ the point at the bottom becomes unstable

From D. Tong's Lecture Notes



For $\omega^2 > g/a$ the minimum at $\psi = 0$ becomes a maximum. The bead will find a new minimum at a finite angle ψ_0. Physically it is the centripetal force which pushes the bead outward.

- Then finally lets determine the period of $\pi/2$ oscillations.

- If the system reaches its maximum at $\pi/2$, the energy is

$$\frac{V_{eff}(\pi/2)}{mag} = \frac{a\omega^2}{2g} = \frac{E}{mag} \equiv \mathcal{E}$$

dimensionless energy

- So

recall $\hat{m} = ma^2$

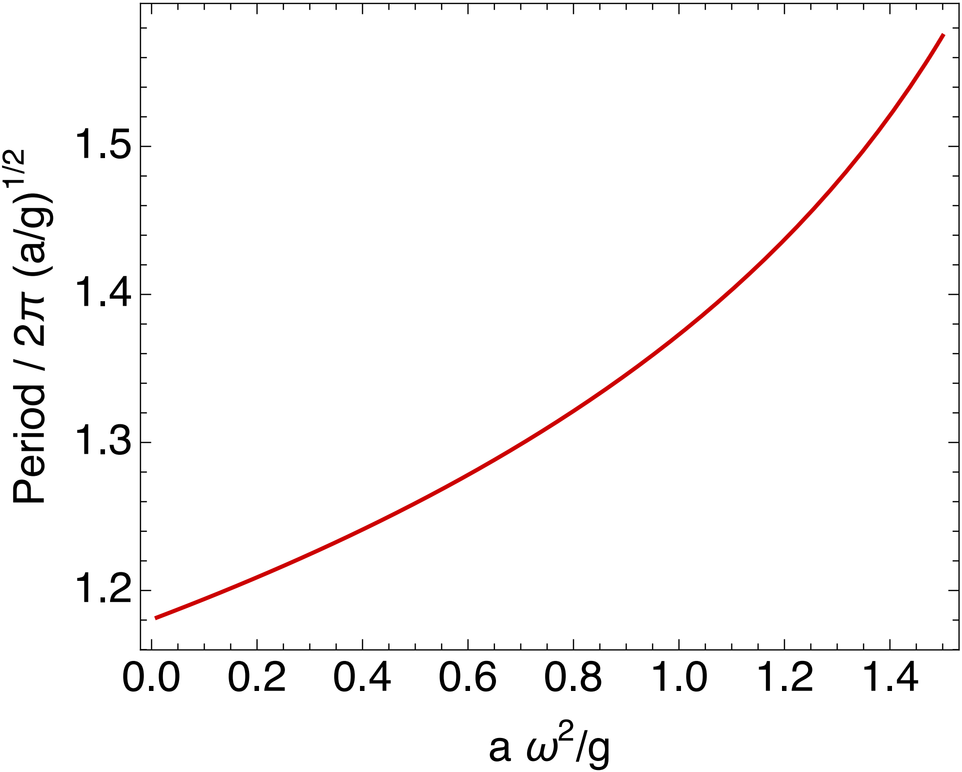
$$\frac{T}{2} = \int_{-\pi/2}^{\pi/2} d\psi \sqrt{\frac{\hat{m}}{2}} \frac{1}{\sqrt{mag}} \frac{1}{(\mathcal{E} - V(\psi))^{1/2}}$$

$$T = \sqrt{\frac{2a}{g}} \int_{-\pi/2}^{\pi/2} d\psi \left(\cos\psi - \left(\frac{a\omega^2}{2g}\right)(1-\sin^2\psi) \right)^{-1/2}$$

- The integral can be done. But it hardly seems worth it. I asked mathematical to plot it

$$T = 2\pi \sqrt{\frac{a}{g}}\, I(a\omega^2/g)$$

$$\text{with} \quad I(u) = \frac{\sqrt{2}}{2\pi} \int_{-\pi/2}^{\pi/2} d\psi \left(\cos\psi - \frac{u}{2}(1-\sin^2\psi)\right)^{-1/2}$$



Period / 2π (a/g)^1/2
1.2
1.3
1.4
1.5
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
a ω^2/g