Alternate Proof of Adiabatic Invariance

- Follows Landau

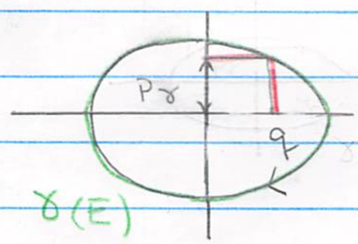
- Consider a 1D system with 1D motion that is periodic and subject to external parameter $\lambda(t)$ wich is slow:

period T ⟶ $T \dfrac{d\lambda}{dt} \ll \lambda$ ⟵ parameter, e.g. $l(t)$ in example

- The Hamiltonian is $H(q, p, \lambda(t))$. If λ is constant, the energy of the system is $E = H(q, p, \lambda)$. Then,

The system makes a periodic orbit in phase space on curve γ, for constant energy. We can, for given $E = H(q,p,\lambda)$, find p implicity:

$$p = p_\gamma(q, E, \lambda)$$



- From Hamilton's EOM, $dH/dt = \partial H/\partial t$:

$$\frac{dE(t)}{dt} = \frac{\partial H}{\partial t} = \left(\frac{\partial H}{\partial \lambda}\right)_{p,q} \frac{d\lambda}{dt}$$

⟵ q is always fixed below

- Now we can average over a time scale Δt which is long compared to the period T, but short compared to the adiabatic time scale, τ_{max}

$$\frac{\Delta E}{\Delta t} = \overline{\left(\frac{\partial H}{\partial \lambda}\right)_{p,q}} \frac{\Delta \lambda}{\Delta t}$$

- Where the average means averaged over a cycle

$$\overline{\left(\frac{\partial H}{\partial \lambda}\right)_{q,p}} = \frac{1}{T} \oint_{\gamma} \left(\frac{\partial H}{\partial \lambda}\right)_{p,q} dt$$

$\leftarrow$ This is the time averaged force exerted by the string on the ring in our example

$$T \equiv \oint dt \equiv \text{period}$$

- rewriting $dt = dq/\dot{q} = dq/(\partial H/\partial p)_\lambda$

$$\frac{\Delta E}{\Delta t} = \frac{\Delta \lambda}{\Delta t} \frac{\oint_{\gamma} \left(\frac{\partial H}{\partial \lambda}\right)_{p} \frac{dq}{\left(\frac{\partial H}{\partial p}\right)_{\lambda}}}{\oint_{\gamma} \frac{dq}{(\partial H/\partial p)_\lambda}}$$

- Now at fixed λ, the energy is constant independent var:

$$\left(\frac{\partial E}{\partial \lambda}\right)_E = 0 = \left(\frac{\partial H}{\partial \lambda}\right)_p \cdot 1 + \left(\frac{\partial H}{\partial p}\right)_\lambda \left(\frac{dp}{d\lambda}\right)_E \qquad p = p_\gamma(q, E, \lambda)$$

or $$(\partial H/\partial \lambda)_p / (\partial H/\partial p)_\lambda = -(\partial p/\partial \lambda)_E$$

So

$$\frac{\overline{\Delta E}}{\Delta t} = \frac{\Delta \lambda}{\Delta t} \frac{\oint -(\partial p/\partial \lambda)_E \, dq}{\oint (\partial p/\partial E)_q \, dq}$$

Re-arranging

$$\oint \underbrace{\left(\frac{\partial p}{\partial E}\right) \frac{\Delta E}{\Delta t} + \frac{\partial p}{\partial \lambda} \frac{\Delta \lambda}{\Delta t}}_{\Delta p/\Delta t} \, dq = 0$$

And thus

inserting 2π is conventional in the classical theory

$$\boxed{\frac{\Delta I}{\Delta t} = 0 \qquad I = \oint p \frac{dq}{2\pi}}$$

During the evolution of the system with a time dependent parameter λ, the Energy will change in time but the adiabatic invariant $I(E, \lambda) \equiv \oint p \, dq/2\pi$ will remain fixed in time.