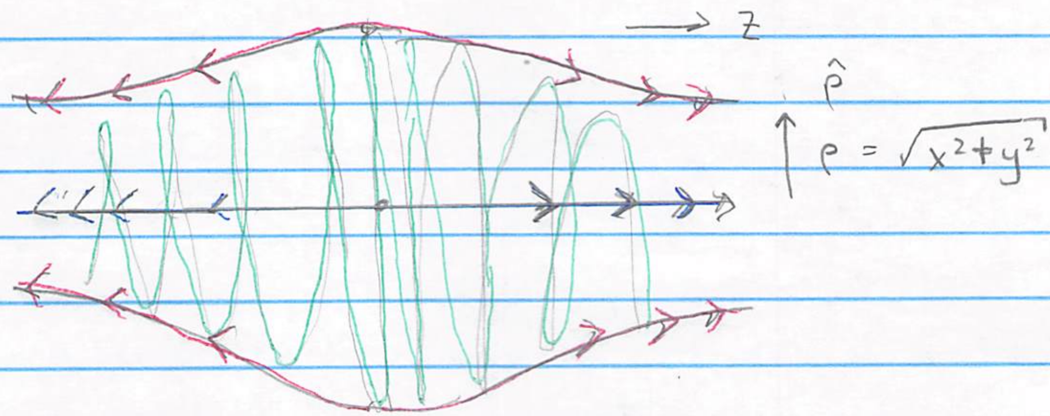
Example - Magnetic Confinement of Charged Particle

- Consider a fast moving electron in a magnetic field which grows slowly along the z-direction, e.g. $B(z) \equiv B_z(z) = B_0(1 + z^2/a^2)$

- Since $\nabla \cdot B = 0$ there is a small correction to $\vec{B} = B_z(z)\hat{z} - \frac{1}{2}B_z'(z)\rho\,\hat{\rho}$ in the directions perpendicular to the z-axis




- As the Charged particle flys toward the region of high field, the transvese (x,y) kinetic energy increases, and the particle's longitudinal kinetic energy decreases until it reaches a stopping point.

## Analysis

- $L = \frac{1}{2} m v^2 + \frac{e}{c} \vec{v} \cdot \vec{A}$

- Now for a constant magnetic field in the z-direction

$$\vec{A} = \frac{B_0}{2}(-y, x, 0) + \text{gradient corrections if } B \text{ is not constant}$$

- The conserved energy (Hamiltonian function)

$$h(q, \dot{q}) = \frac{\partial L}{\partial \vec{v}} \cdot \vec{v} - L = \frac{1}{2} m v^2 = E$$

i.e for $\vec{v} = (\vec{v}_\perp, \dot{z})$

$$E = \frac{1}{2} m \dot{z}^2 + \frac{1}{2} m v_\perp^2$$

The period of orbit is $2\pi/\omega_c$. Recall that for a particle in a magnetic field the "cyclotron" frequency is $\omega_c = eB/mc$. and $\frac{1}{2} m v_\perp^2 = \frac{1}{2} m (\omega_c R)^2$

- Now the particle has small $v_z$. We evaluate the adiabatic invariant for $v_z = 0$, and then recognize that if $z$ and $v_z$ change, the adiabatic invariant will be fixed

$$I = \frac{1}{2\pi} \oint p \cdot dq$$

for circular orbit

- Now for a circular orbit

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = m\vec{v} + \frac{e}{c}\cdot\vec{A}$$

- Now algebra determines the integral invariant for $\dot{z} = 0$

$$I = \frac{1}{2\pi}\oint \underbrace{\left(m\vec{v} + \frac{e}{c}\cdot\vec{A}\right)}_{\vec{p}} \cdot \underbrace{\vec{v}\, dt}_{d\vec{q}}$$

Use the circulation thrm and $\vec{B} = \nabla \times \vec{A}$

$$\Phi = \oint \vec{A}\cdot d\vec{r} = \int \vec{B}\cdot d\vec{a}$$

$$= -B\pi R^2$$

$$I = \frac{1}{2\pi}\oint m v_\perp^2\, dt + \frac{e}{2\pi c}\underbrace{\oint \vec{A}\cdot d\vec{r}}_{\Phi_0 = -B\pi R^2}$$

$$= \frac{1}{2\pi} m v_\perp^2 \frac{2\pi}{\omega_c} - \frac{e}{2\pi c} B\pi R^2$$

Use $\omega_c = eB/mc$

$$\frac{1}{2} m v_\perp^2 = \frac{1}{2} m(\omega_c R)^2$$

$$I = \frac{1}{2} m v_\perp^2 \left(\frac{mc}{eB}\right)$$

so

$$\boxed{\frac{1}{2} m v_\perp^2 = I\,\omega_c \qquad \omega_c = \frac{eB}{mc}}$$

- Now we can use that $I$ is approximately constant as $z$ - slowly changes

$$\frac{1}{2} m \dot{z}^2 + \frac{1}{2} m v_\perp^2 = E$$

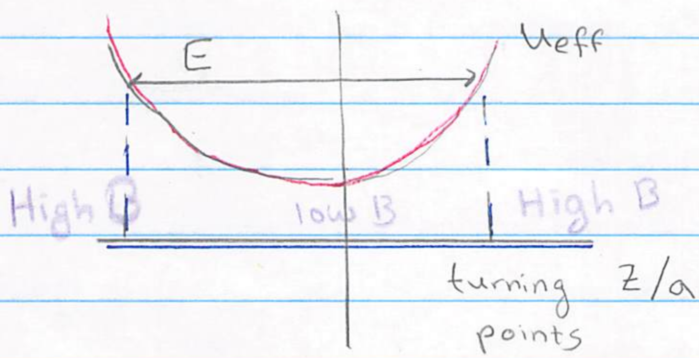
$$\frac{1}{2} m \dot{z}^2 + \frac{I e B(z)}{mc} = E$$

- So for $B(z) = B_o(1 + z^2/a^2)$ we can describe the motion as that of an effective potential

$$\frac{1}{2} m \dot{z}^2 = E - U_{eff}(z)$$

where

$$U_{eff} = \frac{I e B_o}{mc}\left(1 + \frac{z^2}{a^2}\right)$$




which can be used to evaluate the period of oscillations and the turning points