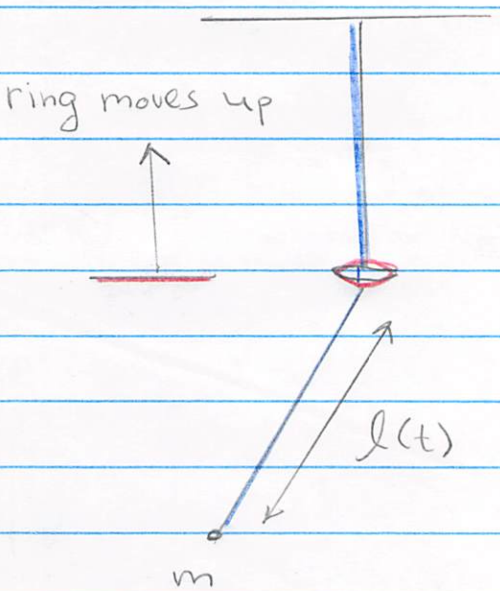
Adiabatic Invariance: Intro

Example:

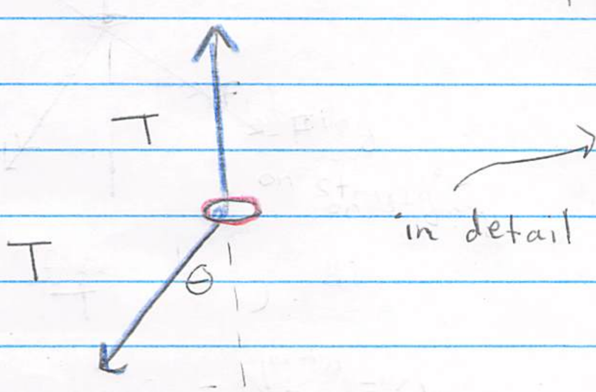


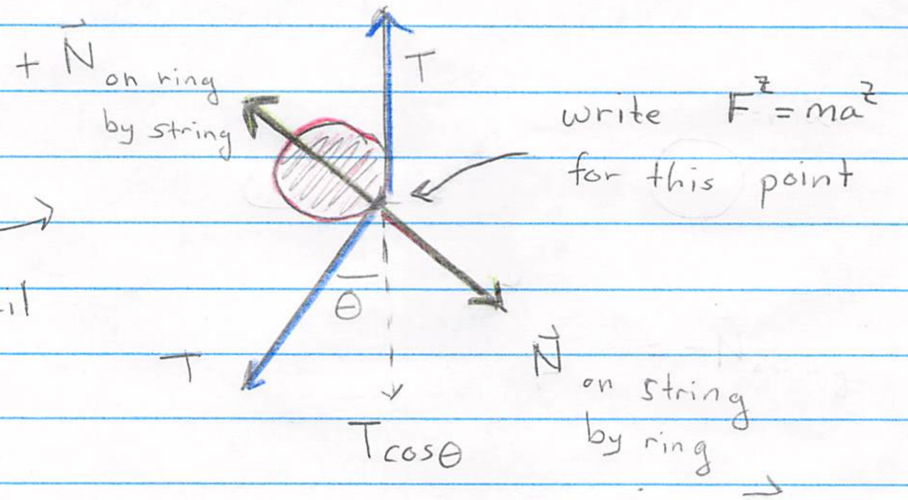
- A string pendulum is passed through a ring. The ring is slowly raised. $\ell(t)$ gets longer.

- The frequency gets slowly lower, $\omega^2 = g/\ell(t)$. The string exerts an upward force on the ring, and as the ring is raised, does work on the ring. The energy available for the oscillations $E(t)$ gets lower. We will see that

$$\boxed{\frac{E(t)}{\omega(t)} \simeq \text{constant}}$$

Simple arguement:





$= -\vec{N}$ on ring by string

We will calculate the work doneby the string on the ring with Newton Laws.
The free-body diagram is shown above. N is the normal forceof the string on the ring (and vice versa).

- Writing the balance of forces (in z) at the pivot:

$$T - T\cos\theta - N_z^{\text{on ring}} = 0$$

$\nwarrow$ z - component of force on ring

- So using $\cos\theta \simeq 1 - \theta^2/2$ and $T \simeq mg$ we have:

$$N_z^{\text{on-ring}} = mg\,\theta^2/2$$

☆ Now lets compute the upward force by averaging over a time, Δt, which is long compared to the period, $T = 2\pi/\omega(t)$, but short compared to the total time T_{max} over which the length and energy have changed appreciably

(1) $$\boxed{2\pi/\omega_0 \ll \Delta t \ll \tau_{max}}$$

Yieding

$$\overline{N}_z \simeq mg\frac{\overline{\theta^2}}{2} = \frac{E(t)}{2\ell}$$

- Thus the work done, $\Delta W = -\Delta E = \overline{N}_z\,\Delta\ell$, implies:

(2) $$-\frac{\Delta E}{\Delta t} = \frac{E}{2\ell}\frac{\Delta\ell}{\Delta t}$$

This equation is easily solved and has

$$E(t)\sqrt{\ell(t)} = \text{const}$$

i.e

$$\boxed{\frac{E(t)}{\omega(t)} = \text{const}}$$

$$\omega^2_{(t)} \propto \frac{g}{\ell(t)}$$

As, we will see in the next section this is a consequence of a general theorem that

$$I = \oint \frac{p\,dq}{2\pi} \leftarrow \text{adiabatic ivariant}$$

is approximately constant under a slow change of a parametr

Adiabatic Invariants

Consider a 1D Hamiltonian which depends on a parameter $\lambda(t)$ and assume that λ changes slowly in time compared to other times $2\pi/\omega_0$ in the problem

$$E(t) \equiv H(q, p, \lambda(t))$$

Over a time Δt which is long compared to $2\pi/\omega_0$ but short compared to τ_{max} (the time over which λ and $E(t)$ change by order one), the energy is approximately constant and the system stays on an iso-energetic contour in phase-space. The area enclosed by this countour (divided by 2π by convention) is

$$I(t) = \int_{\text{Area}} \frac{dp\,dq}{2\pi} = \oint_{\text{1 cycle}} \frac{p\,dq}{2\pi}$$

See figure on next page!

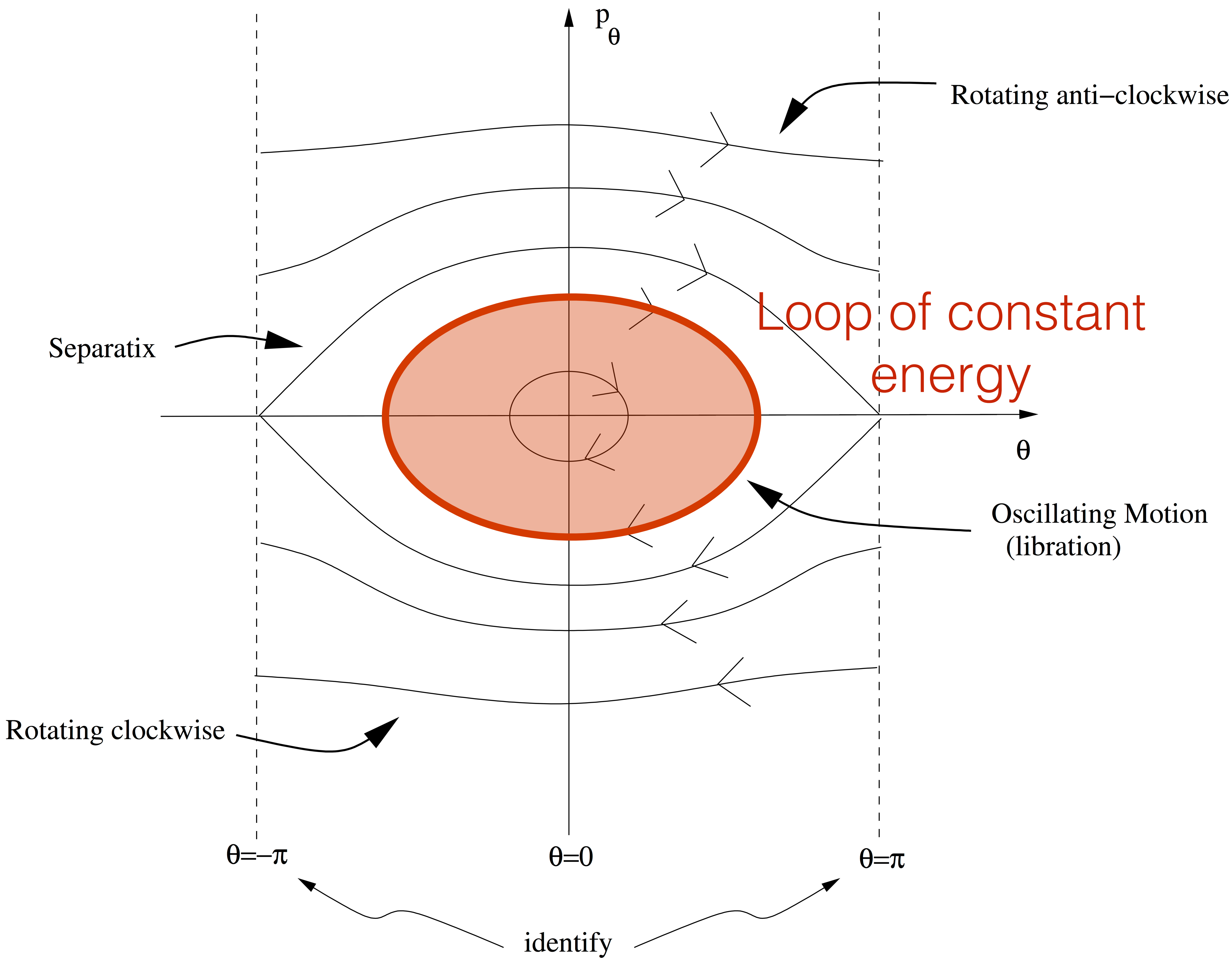
where the integral is done by treating E as constant for the cycle, See below

We are using the curl theorem in 2D, with the vector field $\vec{v} = (p, 0)$. The curl theorem in 2d relates the line integral to the area integral:

$$\oint_{\text{loop}} \vec{v}\cdot \mathrm{d}\vec{\ell} = \int_{\text{area}} d\vec{A}\cdot(\nabla\times\vec{v})\,.$$

We have chosen the loop to circulate in phase-space in a clockwise sense, so that the area points into the page, leading to $\int_{\text{line}} p\,\mathrm{d}q = \int_{\text{area}} \mathrm{d}p\,\mathrm{d}q$. In terms of forms (if you know it), $\mathrm{d}(p\mathrm{d}q) = \mathrm{d}p \wedge \mathrm{d}q$.

$$\text{area} = \int \mathrm{d}p\,\mathrm{d}q = \oint p\,dq$$



- Now we have an <u>exact</u> theorem which says that the area enclosed under the full time evolution will not change in time (Liouville theorem). So we must have over time τ_{max}:

$$\boxed{I(t) = \oint_{\text{cycle}} \frac{p\,dq}{2\pi} \simeq \text{const}}$$

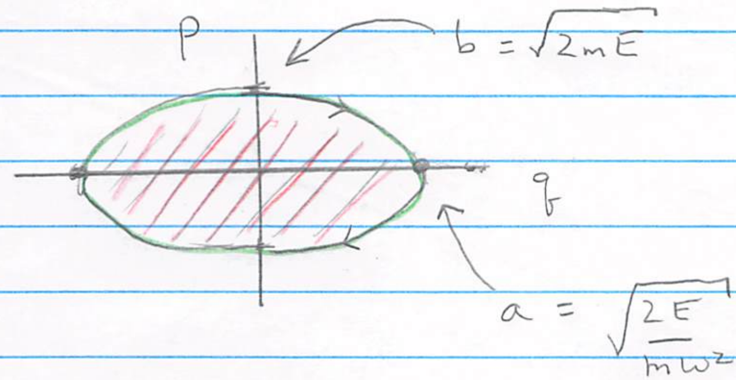
Example SHO

- The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2(t)\, q^2$$

with ω-changing slowly over time τ_{max}. For short times $\Delta t \ll \tau_{max}$ the energy is constant, as is ω

$$E = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 q^2 \leftarrow \text{equation of ellipse: } (x/a)^2 + (y/b)^2 = \text{const}$$



Area of ellipse $= \pi a b$

So

$$I = \oint_{\text{cycle}} \frac{p\,dq}{2\pi} = \int_{\text{area}} \frac{dp\,dq}{2\pi} = \frac{\pi\,ab}{2\pi}$$

$$I = \frac{E}{\omega}$$

Now if E and ω change in time the Lioville theorem guaranities that I is constant i.e.

$$\boxed{I = \frac{E(t)}{\omega(t)} \simeq \text{constant}}$$