The Hamiltonian

- We would like to formulate classical mechanics in terms of coordinates q_A and momenta p_A, instead of coordinates q_A and velocities, $V_A \equiv \dot{q}_A$. We will make a legendre transformation, which we describe below

- First; note
$$dL = \frac{\partial L}{\partial \dot{q}} d\dot{q} + \frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial t} dt$$

$$dL = p\, dV + \underbrace{\frac{\partial L}{\partial q} dq + \frac{\partial L}{\partial t} dt}_{\equiv \text{ spectators}}$$

these variables are not essential to the transformation

- So consider $dH = d(pv - L)$

$$d(pv - L) = v\,dp + p\,dV - (p\,dV + \text{spectators})$$

Important cancelation

So

$$\star \quad dH = v\,dp \underbrace{- \frac{\partial L}{\partial q} dq - \frac{\partial L}{\partial t} dt}_{-(\text{spectators})}$$

In general if q, v, t are our independent variables we would write:

$$dp = \frac{\partial p}{\partial q} dq + \frac{\partial p}{\partial v} dV$$

- But since dp appears naturally it is useful/insightful to take p, q, t instead of v, q, t as the independent variables

- We are allowed to replace v with p provided for each v there is one value of p I.e, given p as a function of v, $p(v) = \frac{\partial L}{\partial v}$, we can invert this equation (at each time and coordinate q) to find v as a function of p, $v(p)$

- So then H as a function of p, q, t is

$$\underline{\underline{H(p,q) = p\, v(p) - L(q, v(p), t)}}$$

☆ Example: $L = \frac{1}{2} m v^2 - U(x)$, $p = \partial L / \partial v = mv$

Inverting $p = mv$ we have $v = p/m$ and thus

$$H = p\left(\frac{p}{m}\right) - \left(\frac{1}{2} m \left(\frac{p}{m}\right)^2 - U(x)\right)$$

$$= \frac{p^2}{2m} + U(q)$$

• Now once we have taken p, q, t as the independent variables, since (see above)

$$dH = v\,dp - \frac{\partial L}{\partial q}\,dq - \frac{\partial L}{\partial t}\,dt$$

Implies also: $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ $\quad \frac{\partial H}{\partial q} = \frac{\partial L}{\partial q}$

So

$$\boxed{\frac{\partial H}{\partial p} = v} \Rightarrow \boxed{\frac{dq}{dt} = \frac{\partial H}{\partial p}}$$

Hamilton Equations

And from the Lagrange equation

$$\frac{d}{dt}p = \frac{\partial L}{\partial q} \Rightarrow \boxed{\frac{dp}{dt} = -\frac{\partial H}{\partial q}}$$

• So the Hamiltonian can be used to determine the evolution of the system. The equations above are Hamiltons EOM and are equivalent to the Euler-Lagrange equations:

For the example:

$$\frac{dx}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \checkmark$$

$$\frac{dp}{dt} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x} \checkmark$$

☆ The hamiltonian is constant in time if H is not an explicit funtion of time

$$\frac{dH}{dt} = \underbrace{\frac{\partial H}{\partial x}\frac{dx}{dt} + \frac{\partial H}{\partial p}\frac{dp}{dt}}_{} + \frac{\partial H}{\partial t}$$

these cancel by the EOM

$$= \frac{\partial H}{\partial t}$$

The Hamiltonian for an important Lagrangian

- This is related to a particle in a magnetic field

$$L = \frac{1}{2} a_{ij}(q)\, \dot{q}^i \dot{q}^j + b_i \dot{q}^i - U(q)$$

↑ symmetric

a, b, U are functions of q

Then

$$p_i = \frac{\partial L}{\partial \dot{q}^i} = a_{ij} \dot{q}^j + b_i$$

Inverting to express $\dot{q}$ in terms of p

$$\star \quad \dot{q}^i = (a^{-1})^{ij} (p_j - b_j)$$

a^{-1} inverse matrix, i.e. $a^{-1} a = \mathbb{1}$

So

$$H = p_i \dot{q}^i - L = p_i \dot{q}^i - \left(\frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j + b_i \dot{q}^i - U \right)$$

$$= (p_i - b_i)\dot{q} - \left(\frac{1}{2} a_{ij} \dot{q}^i \dot{q}^j - U \right)$$

Using $\star$ we have after minor algebra

$$H = \underbrace{\frac{1}{2} (a^{-1})^{ij} (p_i - b_i)(p_j - b_j)}_{\text{Kinetic energy}} + U$$

• Remark:

Some of you might prefer a matrix notation:

$$Q = \begin{pmatrix} q^1 \\ \vdots \\ q^n \end{pmatrix} \qquad P = \begin{pmatrix} p_1 \\ \vdots \\ p_n \end{pmatrix}$$

But try to get over this hangup!

$$A = \begin{pmatrix} a_{ij} \end{pmatrix} \qquad B = \begin{pmatrix} b_1 \\ \vdots \end{pmatrix}$$

• Then

$$L = \frac{1}{2} \dot{Q}^T A \dot{Q} + B^T \dot{Q} - U$$

$$\frac{\partial L}{\partial \dot{Q}} = P = A\dot{Q} + B \implies ☆\ \dot{Q} = A^{-1}(P - B)$$

So

$$H = P^T \dot{Q} - \overbrace{\left(\frac{1}{2} \dot{Q}^T A \dot{Q} + B^T \dot{Q} - U \right)}^{L}$$

$$= (P - B)^T \dot{Q} - \left(\frac{1}{2} \dot{Q}^T A \dot{Q} - U \right)$$

use Eq ☆

$$= \frac{1}{2} (P - B)^T A^{-1} (P - B) + U$$