Poisson Brackets

- Let $f(p,q,t)$ be a function of the coordinates and time
- Then

$$\frac{df}{dt} = \frac{\partial f}{\partial q}\dot{q} + \frac{\partial f}{\partial p}\dot{p} + \frac{\partial f}{\partial t}$$

$$= \underline{\frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial f}{\partial p}\frac{\partial H}{\partial q}} + \frac{\partial f}{\partial t}$$

- The algebraic structure appears frequenty

$$\boxed{\{f, H\} \equiv \frac{\partial f}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial H}{\partial q}\frac{\partial f}{\partial p}}$$

Poisson Bracket

So

$$\boxed{\frac{df}{dt} = \{f, H\} + \frac{\partial f}{\partial t}}$$

- For any two functions f, g with multiple variables

$$\boxed{\{f, g\} = \frac{\partial f}{\partial q^a}\frac{\partial g}{\partial p_a} - \frac{\partial f}{\partial p_a}\frac{\partial g}{\partial q^a}}$$

Properties

- $\{f, g\} = -\{g, f\}$ Antisymmetry
- $\{f_1 + f_2, g\} = \{f_1, g\} + \{f_2, g\}$ Linearity
- Leibnitz rule of Differentiang

$$\{fg, h\} = \{f, h\} g + f\{g, h\}$$

- Jacobi Identity:

$$\{f_1 \{f_2 f_3\}\} + \{f_2 \{f_3 f_1\}\} + \{f_3 \{f_1 f_2\}\} = 0$$

- Can also show e.g.

$$\{q_i q_j\} = 0$$

$$\{p_i p_j\} = 0$$

$$\{q_i p_j\} = \delta_{ij}$$

So

$$\{f, q^i\} = -\frac{\partial f}{\partial p_i} \quad \text{and} \quad \{f, p_i\} = \frac{\partial f}{\partial q^i}$$

Poisson Theorem For Conserved Quantities

Supposed that I, J are conserved quantities by the motion. Then

$$\{I, J\}$$

is also conserved

- Proof:

$$\frac{dI}{dt} = \{I, H\} = 0$$

$$\frac{dJ}{dt} = \{J, H\} = 0$$

- So

$$\frac{d\{I,J\}}{dt} = \{\{I,J\}, H\}$$

$$= -\{H\{IJ\}\}$$

$$= +\{I\cancelto{0}{\{J,H\}}\} + \{J\cancelto{0}{\{H,I\}}\}$$

- Typically the algebra closes after a few steps

Angular Momentum

$$L_1 = r_2 p_3 - r_3 p_2$$

$$L_2 = r_3 p_1 - r_1 p_3$$

So

$$\{L_1, L_2\} = \{r_2 p_3 - r_3 p_2 \,,\, r_3 p_1 - r_1 p_3\}$$

$$= \{r_2 p_3, r_3 p_1\} + \{r_3 p_2, r_1 p_3\}$$

$$- \cancelto{0}{\{r_3 p_2, r_3 p_1\}} - \cancelto{0}{\{r_2 p_3, r_1 p_3\}}$$

These quantities all "commute"

And Similarly (Leibnitz + "differentian" give)

$$\{r_2 p_3, r_3 p_1\} = r_2 \{p_3, r_3 p_1\} + \cancelto{0}{\{r_2, r_3 p_1\}} p_3$$

$$= - r_2 p_1$$

And

$$\{r_3 p_2, r_1 p_3\} = r_3 \cancelto{0}{\{p_2, r_1 p_3\}} + \{r_3, p_1 p_3\} p_2$$

$$= r_1 p_2$$

So

$$\{L_1, L_2\} = r_1 p_2 - r_2 p_1 = L_3$$

Thus we have found a new conserved quantity with Algebra

$$\{L_a, L_b\} = \varepsilon_{abc} L_c$$

The deaper significance of $\vec{L}$ as the generator of rotations will come in what follows.