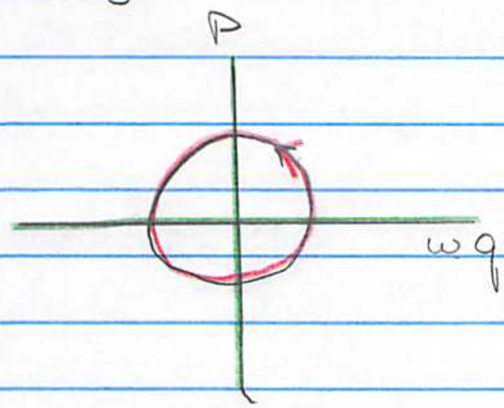
Canonical Transformations

- We want to find new coordinates for phase space.

- Why?



We already discussed that the motion of a SHO is a circle. The motion of a perturbed oscillator is close to a circle. Maybe we should use circular coordinates (Amplitude and Phase)? Or perhaps $a_\pm = \omega q \pm ip$?

- Change of coordinates which preserve the form of Hamiltons Equations are particularly important.

- So we look for a map

$$\left.\begin{aligned} q &\rightarrow Q(q,p) \\ p &\rightarrow P(q,p) \\ h(q,p,t) &\rightarrow \underline{H}(Q,P,t) \end{aligned}\right\}$$

Where the new EOM are:

$$\dot{Q} = \partial \underline{H}/\partial P$$

$$\dot{P} = -\partial \underline{H}/\partial Q$$

- Then the action in the old coordinates is

$$S_1 = \int p\,dq - h\,dt$$

The action in the new coordinates is

$$S_2 = \int P\,dQ - \underline{H}\,dt$$

← the EOM for Q, P are unchange

- The difference in the two actions can only be a total derivative, which modifies the boundary terms without modifying the EOM.

$$S_1 - S_2 = \int_{t_1}^{t_2} \frac{dF}{dt}\,dt$$

Or taking $F(q, Q, t)$ we find

$$\int p\,dq - P\,dQ - (h - \underline{H})\,dt$$

$$= \int \underbrace{\frac{\partial F}{\partial q}\,dq + \frac{\partial F}{\partial Q}\,dQ + \frac{\partial F}{\partial t}\,dt}_{dF/dt}$$

- So we compare both sides

Yielding

$$\boxed{\begin{array}{llll} (1) & p = \dfrac{\partial F}{\partial q}(q, Q, t) & \text{and} & \underline{H} = h + \dfrac{\partial F}{\partial t} \\ (2) & -P = \dfrac{\partial F}{\partial Q}(q, Q, t) & & \end{array}}$$

- Gives a perscription: Use (1) to find $Q(q, P, t)$. then we can evaluate $P(q, Q(q,p,t))$ from (2).

- Rather than working with F it is easier to Legendre transform / integrate by parts

$$S_1 - S_2 = \int p\,dq - P\,dQ - (h - \underline{H})\,dt$$

$$= \int dt \, \frac{dF}{dt}(q, Q, t)$$

- Write $-PdQ = -d(PQ) + QdP$, and bring it to the other side:

$$S_1 - S_2 = \int p\,dq + Q\,dP - (h - \underline{H})\,dt$$

$$= \int \frac{d}{dt}(F + PQ)$$

Then also called F_2

$\underline{\Phi} \equiv F + PQ$ is the Legendre Transform of F

And generates the following canonical map

$$d\Phi(q,P) = p\,dq + Q\,dP - (H - \underline{H})\,dt$$

i.e.

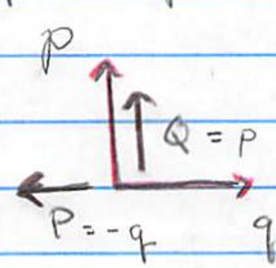
$$\boxed{\begin{array}{lll} (1) & p = \dfrac{\partial \Phi}{\partial q}(q,P) & \underline{H}(Q,P) = h(q,p) + \dfrac{\partial \Phi}{\partial t} \\ (2) & Q = \dfrac{\partial \Phi}{\partial P}(q,P) & \end{array}}$$

- Works like before first we solve for $P(q,p)$ from (1), then evaluate $Q(q,p)$ from (2)

Examples

- $F = qQ$ exchanges q and p

$$p = \frac{\partial F}{\partial q} = Q$$



$$\frac{\partial p}{\partial Q} + \frac{\partial P}{\partial q} = 0$$

$$-P = \frac{\partial F}{\partial Q} = q$$

- Take $\Phi = qP$, this is the identity transformation

$$p = \frac{\partial \Phi}{\partial q} = P \qquad Q = \frac{\partial \Phi}{\partial P} = q$$

- Now that we know the identity transformation we can use a transformation close to the identity

$$\Phi = qP + G(q,P)\lambda$$

small

Then

$$(1) \qquad p = P + \frac{\partial G}{\partial q}(q,P)\lambda$$

$$(2) \qquad Q = q + \frac{\partial G}{\partial P}(q,P)\lambda$$

- Solving (1) by iteration. $p \simeq P$ at zeroth order. Then we can substitute $p = P$ in $\partial G(q,P)/\partial q$ at first order:

$$\boxed{\star \qquad P \simeq p - \frac{\partial G}{\partial q}(q,p)\lambda}$$

$$Q = q + \frac{\partial G}{\partial P}(q,P)\lambda$$

$$\boxed{\star \qquad Q \simeq q + \frac{\partial G}{\partial p}(q,p)\lambda}$$

These two transformation are the infinitessimal transforms discussed first

- Final Example: Scale Transformations

$$\Phi = a\,q\,P$$

So:

$$p = aP \qquad \Longrightarrow \qquad \boxed{\begin{aligned} P &= \frac{p}{a} \\ Q &= aq \end{aligned}}$$

$$Q = aq$$

This is what one expects: if $q \rightarrow Q = aq$, then $p = \frac{\partial L}{\partial \dot{q}}$ should go as $p \rightarrow P = \frac{p}{a}$.

Clearly $dQ\,dP = dq\,dp$ ✓

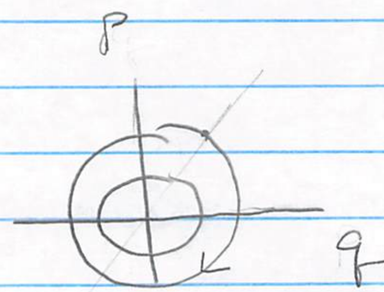
Circlular Coordinates in Phase Space. The SHO.

- The Hamiltonian of the SHO

$$H = \frac{1}{2} p^2 + \frac{1}{2}\omega^2 q^2$$

- Is it possible to find a canonical transform where one of my coordinates P is the Hamiltonian itself? say

$$\tilde{H} = \omega P$$



This is an $(\text{amplitude})^2$ and the phase representation of the Harmonic oscillator. $P = (\text{amplitude})^2$ and Q = phase. In the coordinates the solution is simple:

$$\dot{Q} = \frac{\partial \tilde{H}}{\partial P} = \omega \Rightarrow Q = \omega t + t_0$$

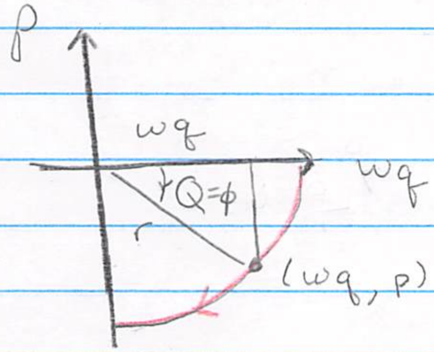
circulating clockwise

$$\dot{P} = -\frac{\partial \tilde{H}}{\partial Q} = 0 \Rightarrow P = \text{constant}$$

A healthy dose of numerology / guess work gives:

$$F(q, Q) = -\frac{\omega q^2}{2} \tan Q$$

- Let de-mystify this generator:



$$\underline{P} \propto (\text{radius})^2 = \frac{1}{2w}\left(p^2 + (wq)^2\right)$$

- Want p and $\underline{P}$, versus q and Q:

$$-\frac{p}{wq} = \tan Q = \tan(\text{phase})$$

$$\underline{P} = \frac{1}{2w}\left(\frac{wq}{\cos Q}\right)^2 \propto \left(\frac{wq}{\cos\phi}\right)^2 \propto (\text{radius})^2$$

- Then we want

$$\left.\begin{aligned} p &= -wq\tan Q = \frac{\partial F}{\partial q} \\ -\underline{P} &= -\frac{1}{2}\frac{wq^2}{\cos^2 Q} = \frac{\partial F}{\partial Q}\end{aligned}\right\} \quad \underline{\underline{\text{or}}} \quad \boxed{F = -\frac{wq^2}{2}\tan Q}$$

- This transformation is "curl free", i.e

$$\frac{\partial p}{\partial Q} = \frac{\partial \underline{P}}{\partial Q} \quad \text{or} \quad \frac{\partial^2 F}{\partial q\,\partial Q} = \frac{\partial^2 F}{\partial Q\,\partial q}$$

which guarantees that the desired /consistent F can be found from $\partial F/\partial q$ and $\partial F/\partial Q$.

- So we can use circular coordinates for phase space

$$Q = \tan\left(\frac{-p}{\omega q}\right)$$

$$P = \frac{1}{2\omega} p^2 + \frac{1}{2\omega} (\omega q)^2$$

The EOM in these coordinates will always be the same $\dot{Q} = \partial H / \partial P$ $\dot{P} = -\partial H / \partial Q$. Using this coordinate system is useful for problems which are close to the SHO.