Problem 1. (MIT/OCW) Spring system on a plane

A massless spring has an unstretched length b and spring constant k, and is used to connect two particles of mass m_1 and m_2. The system rests on a frictionless table and may oscillate, translate, and rotate.

(a) What is the Lagrangian? Write it with two-dimensional cartesian coordinates $\boldsymbol{r}_1 = (x_1, y_1)$ and $\boldsymbol{r}_2 = (x_2, y_2)$. There are four coordinates in total.

(b) Setup a suitable set of generalized coordinates (four in total) to better account for the symmetries of this system. Take one of your coordinates to be $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. What is the Lagrangian in these variables?

(c) Identify three conserved generalized momenta that are associated to cyclic coordinates in the Lagrangian from part (b). If you think you are missing some, try to improve your answer to (b). Briefly explain the physical meaning of each of the three conserved generalized momenta. Show that the equation of motion for r takes the form

$$m_{\text{eff}}\ddot{r} = -\frac{\partial V_{\text{eff}}(r)}{\partial r} \tag{1}$$

with an appropriate m_{eff} and $V_{\text{eff}}(r)$.

(d) Write down the hamiltonian function $h(q, \dot{q}, t)$ for the coordinates chosen in (b), and show that

$$\frac{1}{2}m_{\text{eff}}\dot{r}^2 + V_{\text{eff}}(r) = \text{const} \tag{2}$$

where the const is related to the "internal energy" of the oscillations.

(e) By examining the effective potential and its dependence on the rotation rate, show that there is a solution that rotates but does not oscillate, and discuss what happens to this solution for an increased rate of rotation. (A closed form solution is not necessary. A graphical explanation based on the effective potential will suffice.)

Problem 2. (Goldstein/MIT OCW) Jerky Mechanics

Consider an extension of classical mechanics where the equation of motion involves a triple time derivative, $\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$. Lets use the action principle to derive the corresponding Euler-Lagrange equations. Start with a Lagrangian of the form $L(q^i, \dot{q}^i, \ddot{q}^i, t)$ for n generalized coordinates q^i, and make use of the action principle for paths $q^i(t)$ that have zero variation of both q^i and $\dot{q}^i$ at the end points. Show that

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \frac{\partial L}{\partial q^i} = 0 \tag{3}$$

for each $i = 1 \ldots n$

Problem 3. Equivalent Lagrangians

Each of these is conceptual and requires minimal computation.

(a) (Goldstein) Let $L(q, \dot{q}, t)$ be the Lagrangian for a particle with coordinate q, which satisfies the Euler-Lagrange equations. Show that the Lagrangian

$$L' = L + \frac{dF(q,t)}{dt} \tag{4}$$

yields the same Euler-Lagrange equations as L where F is an arbitrary differentiable function. Give a proof based on and the action principle. We say that L and L' are equivalent. (If you feel like it you might also like to check directly that the EOM are the same.)

(b) (Goldstein) Using the previous problem (Problem 2), what is the equation of motion resulting from

$$L = -\frac{1}{2} m q \ddot{q} - \frac{1}{2}\omega_0^2 q^2 \tag{5}$$

and what is it related to? Explain why this equation of motion is obvious from the Lagrangian in Eq. (5) and the result of part (a).

(c) Consider the action of a free particle

$$S[\boldsymbol{r}(t)] = \int_{t_1}^{t_2} \mathrm{d}t\, C\boldsymbol{v}^2 \tag{6}$$

where $C = m/2$ is a constant normally associated with the mass. Show that the action is unchanged by a Galilean transformation up to boundary terms (i.e. terms that only depend on the coordinates at the endpoints t_1 and t_2). Hence the transformed version gives the same EOM. If the Lagrangian took the form $L = Cv^4$ this would not have been the case. Thus requiring Gallilean invariance fixes the form the velocity dependent action to involve only v^2, and what we call mass is just the coefficient in front of this term.