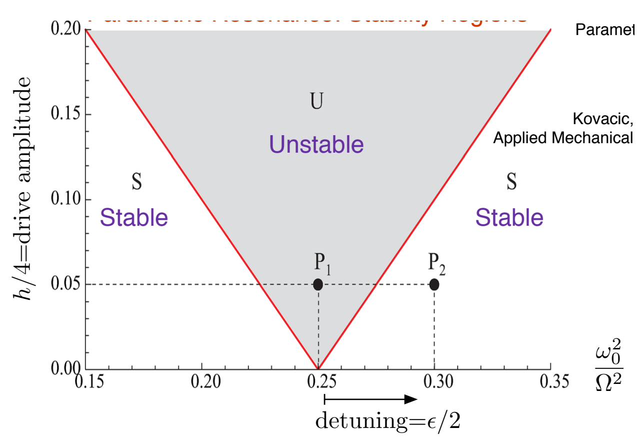
Problem 1. Parametric resonance with damping

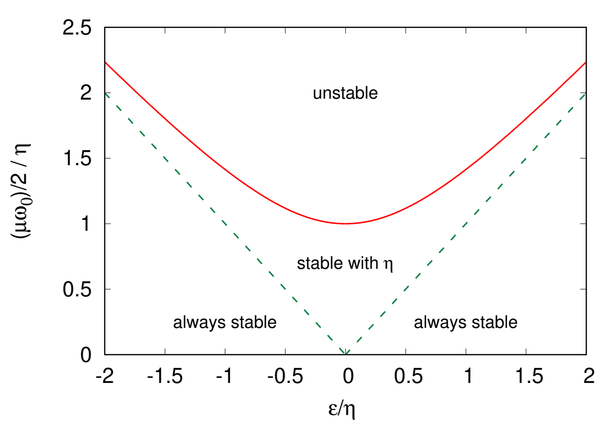
Consider an oscillator with a small damping coefficient η, and a time dependent mass, $m(t) = m_0(1 + \mu\cos(\Omega t))$, with μ is small. The frequency is $\Omega \simeq 2\omega_0 + \epsilon$ with ϵ also small. Thus the equation of motion is

$$\frac{d}{dt}(m(t)\dot{q}) + m_0\omega_0^2 q(t) + m_0\eta\dot{q} = 0 \tag{1}$$

Determine the regions in the ϵ, μ plane where the oscillations are stable and unstable. How is the plot from class (the first plot below) modified by the non-zero damping coefficient?



You should find the following picture:



Problem 2. A pendulum in a harmonic electric field

A simple pendulum consists of a particle of mass m at the end of weightless rod of length ℓ. The particle has a charge q and sits in an electric field of amplitude E_0, directed in the horizontal dirction, which oscillates rapidly with frequency Ω, $\Omega \gg \sqrt{g/\ell}$.

(a) Determine the Lagrangian for this system, and the equation of motion.

(b) Above a critical field strength E_c the position $\phi = 0$ (the bottom of the pendulum) becomes unstable. Determine E_c and determine the new point of stability for $E > E_c$. Sketch the effective potential for $E < E_c$ and $E > E_c$.

(c) Analyze the validity of your approximations. Is the critical field large or small compared to mg/q when your approximation is valid?

Problem 3. (Laurence Yaffe) A driven set of oscillators

General Background: Consider a set of coupled harmonic oscillators interacting with external time dependent forces. The oscillator Lagrangian without the forces reads[1]

$$L_0 = \sum_{ij} \frac{1}{2} M_{ij} \dot{q}^i \dot{q}^j - \frac{1}{2} K_{ij} q^i q^j \,. \tag{2}$$

The Lagrangian for the forces driving the system is

$$L_{\rm int} = \sum_i F_i(t) q^i \,, \tag{3}$$

and the total Lagrangian is $L = L_0 + L_{\rm int}$. As always, switch coordinates to the eigen basis of the generalized eigenvalue problem

$$q^i = \sum_a E^i_a Q^a \,, \tag{4}$$

where the $\vec{E}_a$ is the a-th eigen-vector of the generalized eigenvalue problem, $K\vec{E}_a = \lambda_a M \vec{E}_a$. Recall that the natural frequency assoicated with the a-th normal mode is $\lambda_a = \omega_a^2$, and the eigenvectors are orthonormal with the mass matrix as weight:

$$\sum_{ij} E^i_a M_{ij} E^j_b = \delta_{ab} \,. \tag{5}$$

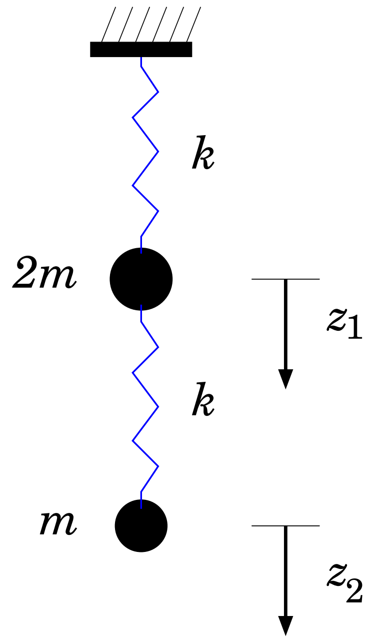
(a) Determine the Lagrangian for the coordinates Q^a, and show that the resulting equation of motion is

$$\ddot{Q}^a + \omega_a^2 Q^a = F_a \,, \tag{6}$$

where $F_a = \sum_i F_i E^i_a$ is the projection of the force vector $\vec{F} = (F_i)$ onto the a-th normal mode, i.e. $F_a = \vec{F} \cdot \vec{E}_a$.

Problem: Now consider two masses, $m_1 = 2m$ and $m_2 = m$, are suspended in a uniform gravitational field g by identical massless springs with spring constant k. Assume that only vertical motion occurs, and let z_1 and z_2 denote the vertical displacement of the masses from their equilibrium positions, increasing in the downward direction as shown. An external time-dependent force $F(t)$ is applied to the lower mass (with $F > 0$ indicating a downward vertical force). Assume that the external force vanishes as $t \to \pm\infty$, with the system initially at rest in its equilibrium configuration at time $-\infty$. Let $F(\omega)$ denote the Fourier transform of $F(t)$.

[1] For the rest of this problem we will not use the summation convention.



(b) Construct the Lagrangian for the system without the force and find the normal modes and frequencies. Then include the external force, and find the resulting equations of motion.

(c) Solve for the motion of both masses (expressed as an integral involving the time-dependent force).

(d) Find the total work done on the system by the external force, $W = E(+\infty) - E(-\infty)$. Show that it can be expressed in the form

$$W = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) |F(\omega)|^2 \tag{7}$$

with $\chi(\omega)$ real and positive. $\chi(\omega)$ is known as the response function, and will be proportional to a delta-functions in the absence of damping.

(e) If a small damping term is added to each equation of motion, so $m_i \ddot{z}_i \to m_i \ddot{z}_i + m_i \eta \dot{z}_i$, make an educated guess how this qualitatively changes the response function $\chi(\omega)$ and make a sketch of $\chi(\omega)$.

(f) (Optional) With the dissipation described in the previous item, again find the total work done by the force on the system. (W is not equal to $E(\infty) - E(-\infty)$, since the work done is ultimately dissipated away.) Determine $\chi(\omega)$ in this case.