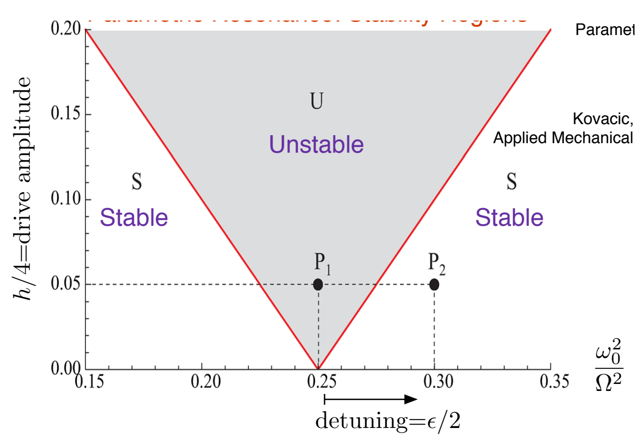
Problem 1. Parametric resonance with damping

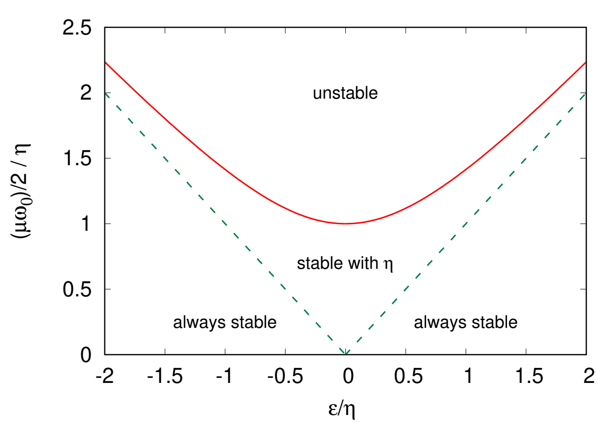
Consider an oscillator with a small damping coefficient η, and a time dependent mass, $m(t) = m_0(1 + \mu\cos(\Omega t))$, with μ is small. The frequency is $\Omega \simeq 2\omega_0 + \epsilon$ with ϵ also small. Thus the equation of motion is

$$\frac{d}{dt}(m(t)\dot{q}) + m_0\omega_0^2 q(t) + m_0\eta\dot{q} = 0 \tag{1}$$

Determine the regions in the ϵ, μ plane where the oscillations are stable and unstable. How is the plot from class (the first plot below) modified by the non-zero damping coefficient?



You should find the following picture:



Solution:

(a) The equation of motion takes the form

$$m(t)\ddot{q} + m_0\omega_0^2 q(t) + m_0\eta\dot{q} + \dot{m}\dot{q} = 0 \tag{2}$$

we may divide by $m(t)$ yielding in a first approximation

$$\ddot{q} + \omega_0^2 q(t) - \mu\omega_0^2\cos(\Omega t)q(t) + \eta\dot{q} - \mu\Omega\sin(\Omega t)\dot{q} = 0\,. \tag{3}$$

We are keeping terms linear in μ and η expanding

$$\frac{m_0\omega_0^2}{m(t)} \simeq \omega_0^2 - \mu\omega_0^2\cos(\Omega t)\,. \tag{4}$$

We write

$$\ddot{q} + \omega^2 q + (-\omega^2 + \omega_0^2)q(t) - \mu\omega_0^2\cos(\Omega t)q(t) + \eta\dot{q} - \mu\Omega\sin(\Omega t)\dot{q} = 0\,. \tag{5}$$

Treating the detuning term as a perturbation

$$(-\omega^2 + \omega_0^2) \simeq -2\omega_0\Delta\omega = -\omega_0\epsilon\,. \tag{6}$$

Some trig identities will be needed below

$$\cos(2x)\cos(x) = \frac{1}{2}\cos(3x) + \frac{1}{2}\cos(x) \tag{7}$$

$$\cos(2x)\sin(x) = \frac{1}{2}\sin(3x) - \frac{1}{2}\sin(x) \tag{8}$$

$$\sin(2x)\cos(x) = \frac{1}{2}\sin(3x) + \frac{1}{2}\sin(x) \tag{9}$$

$$\sin(2x)\sin(x) = -\frac{1}{2}\cos(3x) + \frac{1}{2}\cos(x) \tag{10}$$

For a harmonic oscillator driven with frequency ω we have

$$q(t) = A\cos(\omega t) + B\sin(\omega t) \tag{11}$$

In this case the parametric resonance essentially drives the frequency at $\omega = \Omega/2$ as we will see below. Heuristically this is because the (nearly resonant) driving term is

$$\cos(2\omega t)\cos(\omega t) = \frac{1}{2}\cos(3\omega) + \underbrace{\frac{1}{2}\cos(\omega t)}_{\text{nearly on resonance}} \tag{12}$$

In a rotating wave approximation we allow the A and B to depend on time

$$q^{(0)}(t) = A(t)\cos(\omega t) + B(t)\sin(\omega t) \tag{13}$$

and adjust A and B to remove the secular divergence. Substituting this form into the equations of motion using that

$$\ddot{q} \simeq -\omega^2 q^{(0)} + 2\dot{B}\omega\cos(\omega t) - 2\dot{A}\omega\sin(\omega t) \tag{14}$$

Since the terms involving $\dot{q}$ are already small (proportional to η and μ) we may take simply

$$\eta\dot{q} \simeq B\eta\omega\cos(\omega t) - A\eta\omega\sin(\omega t) \tag{15}$$

Tackling the first non-linear term

$$\begin{aligned} -\mu\omega_0^2\cos(\Omega t)(A(t)\cos(\omega t) + B\sin(\omega t)) = &-\frac{\mu\omega_0^2 A}{2}\left(\cos(\Omega+\omega) + \cos(\Omega-\omega_0)\right) \\ &+ \frac{\mu\omega_0^2 B}{2}\left(\sin(\Omega+\omega) + \sin(\Omega-\omega)\right) \end{aligned} \tag{16}$$

Now $\Omega = 2\omega = 2\omega_0 + \epsilon$ with $\epsilon \ll 1$. So the terms which are (approximately) on resonance, are the terms with $\Omega - \omega = \omega \simeq \omega_0$. Neglecting the $\Omega + \omega$ terms as being small, and away from resonance, we have

$$-\mu\omega_0^2\cos(\Omega t)(A(t)\cos(\omega t) + B\sin(\omega t)) \simeq -\frac{\mu\omega_0^2 A}{2}\cos(\omega t) + \frac{\mu\omega_0^2 B}{2}\sin(\omega t)) \,. \tag{17}$$

Similarly, we may treat treat the second nonlinear term, dropping terms which are proportional to $\Omega + \omega$, but keeping the nearly resonant terms as before we find

$$\begin{aligned} -\mu\Omega\sin(\Omega t)(-A\omega\sin(\omega t) + B\omega\cos(\omega t)) =& \frac{A\mu\Omega\omega}{2}\cos(\omega t) - \frac{B\mu\omega\Omega}{2}\sin\omega && (18) \\ =& A\mu\omega^2\cos(\omega t) - B\mu\omega^2\sin\omega \,. && (19) \end{aligned}$$

Finally we have the detuning term

$$(-\omega^2 + \omega_0^2)q = -\omega\epsilon A\cos(\omega t) - \omega\epsilon B\sin(\omega t) \tag{20}$$

Collecting the coefficients of $\sin(\omega t)$ and $\cos(\omega t)$ yields

$$\begin{aligned} \ddot{q}^{(1)} + \omega^2 q^{(1)} + &\left[2\dot{B}\omega + B\eta\omega - \frac{\mu\omega^2}{2}A + A\mu\omega^2 - \omega\epsilon A\right]\cos(\omega t) \\ &+ \left[-2\dot{A}\omega - A\eta\omega + \frac{\mu\omega^2}{2}B - B\mu\omega^2 - \omega\epsilon B\right]\sin(\omega t) \\ &+ \text{neglected } 3\omega \text{ terms} = 0 \end{aligned} \tag{21}$$

Requiring that the secular terms vanish yields and neglecting (in a first approximation) the distinction between ω and ω_0

$$\frac{d}{dt}\begin{pmatrix} A \\ B \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \eta & \frac{\mu\omega_0}{2} + \epsilon \\ \frac{\mu\omega_0}{2} - \epsilon & \eta \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} \,. \tag{22}$$

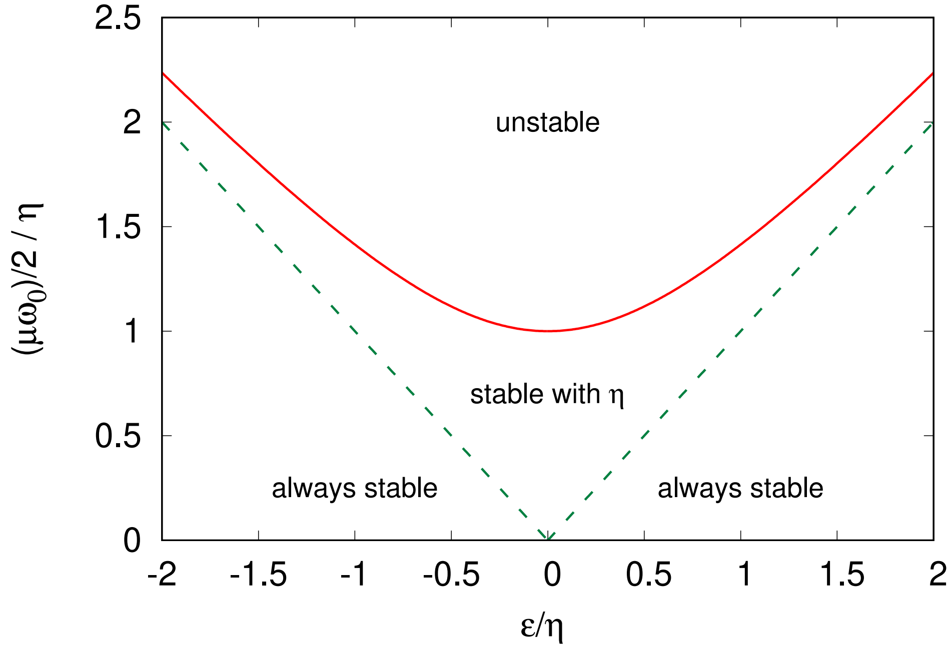


Figure 1: Stability regions

Comparison with the course notes shows that μ plays the role of h. The eigenvalues of the matrix, $\dot{A} \sim e^{(\lambda/2)t}$

$$\lambda_{\pm} = \left[\pm\sqrt{(\mu\omega_0/2)^2 - \epsilon^2} - \eta\right] \tag{23}$$

If an eigenvalue has a positive real part then the motion will grow exponentially and be unstable; the solutions are proportional to $e^{(\lambda/2)t}$. This can happen if the drive parameter μ is sufficiently large. Neglecting the dissipation, this happens when $\mu\omega_0/2 > \epsilon$ as discussed in class. In the current case (with dissipation), the drive amplitude must overcome the $-\eta$ term. We can determine the boundary of stability by setting λ_+ to zero:

$$\sqrt{(\mu\omega_0/2)^2 - \epsilon^2} - \eta = 0\,. \tag{24}$$

So, the motion will be unstable when

$$\left(\frac{\mu\omega_0}{2}\right)^2 > \eta^2 - \epsilon^2\,. \tag{25}$$

This is the equation of a hyperbola which is shown in Fig. 1.

Problem 2. A pendulum in a harmonic electric field

A simple pendulum consists of a particle of mass m at the end of weightless rod of length ℓ. The particle has a charge q and sits in an electric field of amplitude E_0, directed in the horizontal dirction, which oscillates rapidly with frequency Ω, $\Omega \gg \sqrt{g/\ell}$.

(a) Determine the Lagrangian for this system, and the equation of motion.

(b) Above a critical field strength E_c the position $\phi = 0$ (the bottom of the pendulum) becomes unstable. Determine E_c and determine the new point of stability for $E > E_c$. Sketch the effective potential for $E < E_c$ and $E > E_c$.

(c) Analyze the validity of your approximations. Is the critical field large or small compared to mg/q when your approximation is valid?

Solution:

(a) Force on the particle is

$$F_x(t) = qE_0 \cos(\Omega t) \tag{26}$$

which gives a Lagrangian

$$L_E = qE_0 \cos(\Omega t) x(t) = qE_0 \cos(\Omega t) \ell \sin(\phi) \tag{27}$$

So the Lagrangian of the system is ultimately

$$L = \frac{1}{2} m\ell^2 \dot{\phi}^2 - mg\ell(1 - \cos(\phi)) + qE_0 \cos(\Omega t) \ell \sin(\phi) \tag{28}$$

Then the equation of motion is

$$m\ell^2 \ddot{\phi} = -mg\ell \sin\phi + qE_0 \cos(\Omega t) \ell \cos(\phi) \tag{29}$$

(b) Then, using the pondermotive time approximate discussed which is valid for large Ω, we write $\phi(t) = \Phi(t) + \xi$, and then at lowest order find

$$m\ell^2 \ddot{\xi} = qE_0 \ell \cos(\Omega t) \cos(\Phi) \tag{30}$$

Or solving for ξ we find

$$\xi(t) = -\frac{qE_0 \ell \cos(\Omega t) \cos(\Phi)}{(m\ell^2)\Omega^2} \,. \tag{31}$$

At next order there is an additional torque on the slow variable Φ

$$\begin{aligned}
\overline{qE_0\ell \cos(\Omega t)\cos(\Phi + \xi)} \simeq& \overline{qE_0\ell\cos(\Omega t)\left[\cos(\Phi) - \sin(\Phi)\xi(t)\right]} && (32)\\
=& \frac{(qE_0\ell)^2}{(m\ell^2)\Omega^2}(\sin\Phi\cos\Phi)\overline{\cos^2(\Omega t)} && (33)\\
=& \frac{(qE_0\ell)^2}{2(m\ell^2)\Omega^2}(\sin\Phi\cos\Phi) && (34)\\
=& -\frac{(qE_0\ell)^2}{4(m\ell^2)\Omega^2}\frac{\partial(\cos^2\Phi)}{\partial\Phi} && (35)
\end{aligned}$$

With this the equation of motion for the slow variable is

$$m\ell^2 \ddot{\Phi} = -\frac{\partial}{\partial \Phi}\left[-mg\ell\cos(\Phi) + \frac{(qE_0\ell)^2}{4m\ell^2\Omega^2}\cos^2\Phi\right] \tag{36}$$

So the potential takes the form

$$\frac{U_{\text{eff}}}{mg\ell} = -\cos(\Phi) + \alpha\cos^2(\Phi) \tag{37}$$

with

$$\alpha \equiv \frac{(qE_0)^2}{(2mg)^2}\left(\frac{\omega_0}{\Omega}\right)^2 \qquad \text{with} \qquad \omega_0^2 = \frac{g}{\ell} \tag{38}$$

The parameter α is the ratio of forces due to the electric field and gravity, multiplied by the ratio of the the two frequencies in this problem ω_0/Ω. We have assumed here that $\Omega \gg \omega_0$. So α is of order unity only when the electric field is large compared to gravity $qE \gg mg$.

Then we may expand the potential for small Φ

$$\frac{U_{\rm eff}}{mg\ell} = 1 + \alpha + \left(-\frac{\Phi^2}{2} + \alpha\,\Phi^2\right) \tag{39}$$

So we see that for

$$\alpha > \frac{1}{2} \tag{40}$$

the potential will not have a minimum at $\Phi = 0$.

We can find the new minimum. By differentiation

$$\frac{\partial}{\partial \cos\Phi}\left(\frac{U_{\rm eff}}{mg\ell}\right) = -1 + 2\alpha\cos(\Phi) = 0\,. \tag{41}$$

So the point of stability is

$$\cos\Phi_* = \frac{1}{2\alpha}\,. \tag{42}$$

The potential as function of $\cos(\Phi)$ is sketched on the next page

(c) The parameter α is of order unity when the field is large enough $E > E_c$. Since we have assumed that $\omega_0 \ll \Omega$, this can only happen for $qE_0 \gg mg$.

Problem 3. (Laurence Yaffe) A driven set of oscillators

General Background: Consider a set of coupled harmonic oscillators interacting with external time dependent forces. The oscillator Lagrangian without the forces reads[1]

$$L_0 = \sum_{ij} \frac{1}{2} M_{ij} \dot{q}^i \dot{q}^j - \frac{1}{2} K_{ij} q^i q^j \,. \tag{43}$$

The Lagrangian for the forces driving the system is

$$L_{\rm int} = \sum_i F_i(t) q^i \,, \tag{44}$$

and the total Lagrangian is $L = L_0 + L_{\rm int}$. As always, switch coordinates to the eigen basis of the generalized eigenvalue problem

$$q^i = \sum_a E^i_a Q^a \,, \tag{45}$$

where the $\vec{E}_a$ is the a-th eigen-vector of the generalized eigenvalue problem, $K\vec{E}_a = \lambda_a M \vec{E}_a$. Recall that the natural frequency assoicated with the a-th normal mode is $\lambda_a = \omega_a^2$, and the eigenvectors are orthonormal with the mass matrix as weight:

$$\sum_{ij} E^i_a M_{ij} E^j_b = \delta_{ab} \,. \tag{46}$$

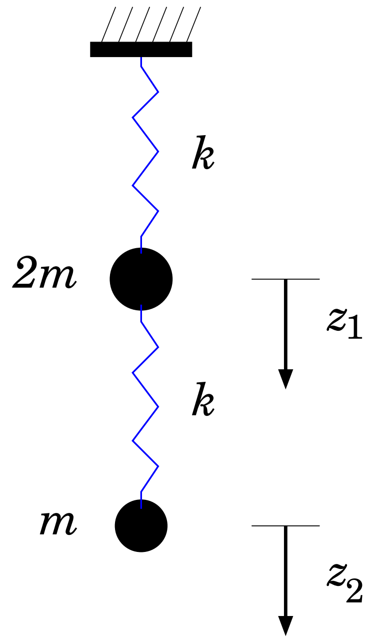
(a) Determine the Lagrangian for the coordinates Q^a, and show that the resulting equation of motion is

$$\ddot{Q}^a + \omega_a^2 Q^a = F_a \,, \tag{47}$$

where $F_a = \sum_i F_i E^i_a$ is the projection of the force vector $\vec{F} = (F_i)$ onto the a-th normal mode, i.e. $F_a = \vec{F} \cdot \vec{E}_a$.

Problem: Now consider two masses, $m_1 = 2m$ and $m_2 = m$, are suspended in a uniform gravitational field g by identical massless springs with spring constant k. Assume that only vertical motion occurs, and let z_1 and z_2 denote the vertical displacement of the masses from their equilibrium positions, increasing in the downward direction as shown. An external time-dependent force $F(t)$ is applied to the lower mass (with $F > 0$ indicating a downward vertical force). Assume that the external force vanishes as $t \to \pm\infty$, with the system initially at rest in its equilibrium configuration at time $-\infty$. Let $F(\omega)$ denote the Fourier transform of $F(t)$.

[1] For the rest of this problem we will not use the summation convention.



(b) Construct the Lagrangian for the system without the force and find the normal modes and frequencies. Then include the external force, and find the resulting equations of motion.

(c) Solve for the motion of both masses (expressed as an integral involving the time-dependent force).

(d) Find the total work done on the system by the external force, $W = E(+\infty) - E(-\infty)$. Show that it can be expressed in the form

$$W = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \chi(\omega) |F(\omega)|^2 \tag{48}$$

with $\chi(\omega)$ real and positive. $\chi(\omega)$ is known as the response function, and will be proportional to a delta-functions in the absence of damping.

(e) If a small damping term is added to each equation of motion, so $m_i \ddot{z}_i \to m_i \ddot{z}_i + m_i \eta \dot{z}_i$, make an educated guess how this qualitatively changes the response function $\chi(\omega)$ and make a sketch of $\chi(\omega)$.

(f) (Optional) With the dissipation described in the previous item, again find the total work done by the force on the system. (W is not equal to $E(\infty) - E(-\infty)$, since the work done is ultimately dissipated away.) Determine $\chi(\omega)$ in this case.

(a) The kinetic term is

$$T = \frac{1}{2}(2m)\dot{z}_1^2 + \frac{1}{2}m\dot{z}_2^2 \tag{49}$$

The potential term is

$$U = \frac{1}{2}kz_1^2 + \frac{1}{2}k(z_2 - z_1)^2 \tag{50}$$

While the force term adds

$$L = F(t)z_2 \tag{51}$$

Differentiation of $L = T - U + L_F$ gives

$$\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix} \begin{pmatrix} \ddot{z}_1 \\ \ddot{z}_2 \end{pmatrix} = \begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} 0 \\ F(t) \end{pmatrix} \tag{52}$$

(b) Let us look for some normal modes. We find the eigenvectors of the eigenvalue problem

$$\omega^2 \underbrace{\begin{pmatrix} 2m & 0 \\ 0 & m \end{pmatrix}}_{\mathcal{M}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underbrace{\begin{pmatrix} 2k & -k \\ -k & k \end{pmatrix}}_{\mathcal{K}} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{53}$$

which leads to

$$\omega_{\pm} = \frac{k}{2m}(2 \pm \sqrt{2}) \tag{54}$$

with eigenmodes

$$E_{\pm} = \begin{pmatrix} \pm\sqrt{2} \\ 2 \end{pmatrix} \tag{55}$$

which satisfy

$$\mathcal{K}E_{\pm} = -\omega^2 \mathcal{M} E_{\pm} \tag{56}$$

The solution is expanded in terms of this basis

$$\begin{pmatrix} z_1(t) \\ z_2(t) \end{pmatrix} = Z_+(t)E_+ + Z_-(t)E_- \tag{57}$$

In the eigen-mode bases we can expand the force

$$\begin{pmatrix} 0 \\ F(t) \end{pmatrix} = \mathcal{M}\mathcal{M}^{-1} \begin{pmatrix} 0 \\ F(t) \end{pmatrix} = \mathcal{M} \begin{pmatrix} 0 \\ \frac{F(t)}{m} \end{pmatrix} = \mathcal{M} \left[\frac{F(t)}{4m} E_+ + \frac{F(t)}{4m} E_- \right] \tag{58}$$

Then the equation of motion for Z_+ and Z_- read

$$\mathcal{M} \left[\left(\ddot{Z}_+ + \omega_+^2 Z_+ + \frac{F(t)}{4m} \right) E_+ + \left(\ddot{Z}_- + \omega_-^2 + \frac{F(t)}{4m} \right) E_- \right] = 0 \tag{59}$$

We note that the vectors are orthogonal with $\mathcal{M}$ as a weight:

$$E_+^T \mathcal{M} E_- = 0\,, \tag{60}$$

implying that

$$\ddot{Z}_+ + \omega_+^2 Z_+ = \frac{F(t)}{4m} \tag{61a}$$

$$\ddot{Z}_- + \omega_-^2 Z_- = \frac{F(t)}{4m} \tag{61b}$$

These equations are solved directly using the green function

$$Z_\pm(t) = \int_{-\infty}^{\infty} dt_0 G_R(t - t_0) \frac{F(t_0)}{4m} \tag{62}$$

where

$$G_R(t, t_0) = \theta(t - t_0) \frac{\sin(\omega_\pm(t - t_0)}{\omega_\pm} \tag{63}$$

(c) Then the total work on the system is

$$W = \int_{-\infty}^{\infty} dt F(t) \dot{z}_2(t) \tag{64}$$

yielding

$$W = \int_{-\infty}^{\infty} dt F(t) \left(2\dot{Z}_+ + 2\dot{Z}_-\right) \tag{65}$$

We worked this out in the previous homework. For a single oscillator with a force $F(t)$

$$m\ddot{x} + m\omega_0^2 x = F(t) \tag{66}$$

The energy is

$$E(t) = \frac{m}{2}|a(t)|^2 = \frac{1}{2m} \left| \int_{-\infty}^{\infty} dt_0 e^{-i\omega_0 t_0} F(t_0) \right|^2 = \frac{1}{2m} |F(\omega_0)|^2 \tag{67}$$

The problem is almost identical here, but we should replace

$$F(t) \to \frac{F(t)}{4} \tag{68}$$

as shown by the equation of motion in Eq. (61a).

$$W = \int_{-\infty}^{\infty} dt \, 8 \times \left[\frac{F(t)}{4} \dot{Z}_+ + \frac{F(t)}{4} \dot{Z}_- \right] \tag{69}$$

$$= 8 \frac{1}{2m\,(16)} \left[|F(\omega_+)|^2 + |F(\omega_-)|^2 \right] \tag{70}$$

$$= \frac{1}{4m} \left[|F(\omega_+)|^2 + |F(\omega_-)|^2 \right] \tag{71}$$

So the response function is

$$\chi(\omega) = \frac{1}{8m} \left[2\pi\delta(\omega - \omega_+) + 2\pi\delta(\omega - \omega_-) + 2\pi\delta(\omega + \omega_+) + 2\pi\delta(\omega + \omega_-) \right] \tag{72}$$

(d) If the damping is small each of the δ-functions of the previous part is smeared. We may proceed a bit differently.

$$W = \int_{-\infty}^{\infty} dt F(t) \left(2\dot{Z}_+ + 2\dot{Z}_- \right) \tag{73}$$

$$W = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt_0\, F(t) \left(2\, \partial_t G_R(t - t_0; \omega_+) \frac{F(t_0)}{4m} + (\omega_+ \to \omega_-) \right) \tag{74}$$

Now we use the properties of Fourier transformations to express the result in Fourier space. We use that

$$\int dt A(t)\, \partial_t B(t) = \int \frac{d\omega}{2\pi} A(-\omega) \left(-i\omega B(\omega) \right) \tag{75}$$

with $A(t) = F(t)$ and

$$\partial_t B(t) = \int_{-\infty}^{\infty} dt_0\, \partial_t G_R(t - t_0; \omega_+) \frac{F(t_0)}{2m} \,. \tag{76}$$

We have also haved used the fact that each time derivative simply adds a $(-i\omega)$ in Fourier space, i.e. the following are Fourier transform pairs:

$$\int dt e^{i\omega t}\, \partial_t^n B(t) = (-i\omega)^n B(\omega) \,. \tag{77}$$

We then use that the Fourier transform of a convolution is the product of Fourier transforms, i.e. if

$$B(t) = \int_{-\infty}^{\infty} dt_0\, C(t - t_0)\, D(t_0) \,, \tag{78}$$

Then

$$B(\omega) = \int dt\, e^{i\omega t}\, B(t) = C(\omega) D(\omega) \,. \tag{79}$$

If these properties of Fourier transforms are not familiar take charge and learn about them on your own. We have finally

$$\text{Fourier Transform of Eq. 76} = -i\omega B(\omega) = -i\omega G_R(\omega; \omega_+) \frac{F(\omega)}{2m} \,. \tag{80}$$

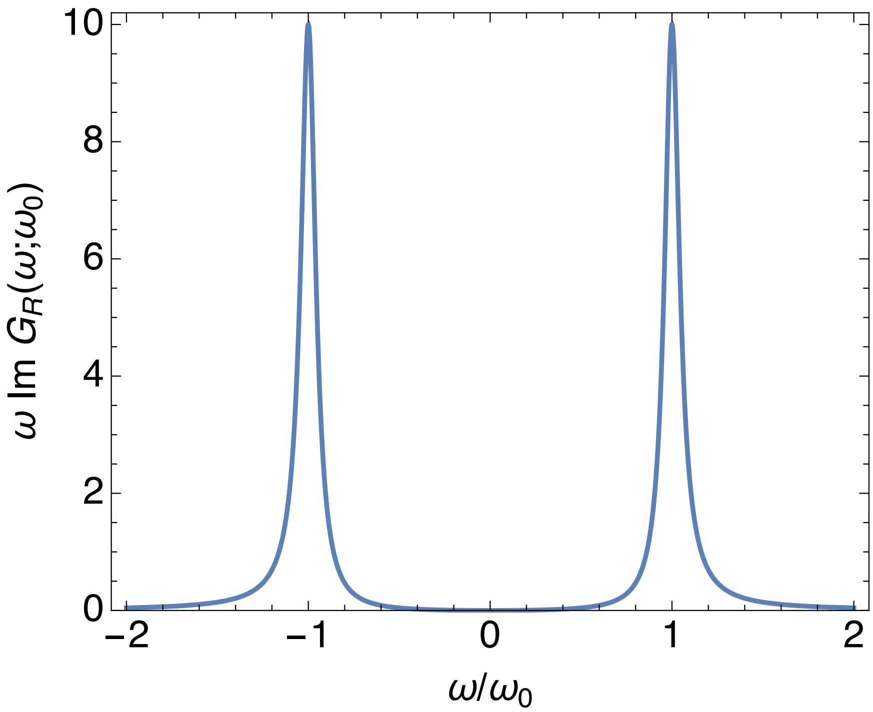
and thus we have

$$W = \frac{1}{2m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, F(-\omega) (-i\omega G_R(\omega; \omega_+)) F(\omega) + (\omega_+ \to \omega_-) \tag{81}$$

The retarded Green function in Fourier space was worked out in class

$$G_R(\omega; \omega_+) = \frac{1}{-\omega^2 + \omega_+^2 - i\omega\eta} \tag{82}$$

$$= \frac{(-\omega^2 + \omega_+^2)}{(\omega^2 - \omega_+^2)^2 + (\omega\eta)^2} + \frac{i\omega\eta}{(\omega^2 - \omega_+^2)^2 + (\omega\eta)^2} \tag{83}$$



The real part of $G_R(\omega)$ is even, while the imaginary part of $G_R(\omega)$ is odd. The imaginary part of the retarded Green function discussed in class is

$$\omega \mathrm{Im} G_R(\omega;\omega_0) = \frac{\omega^2 \eta}{(-\omega^2+\omega_0^2)^2 + (\omega\eta)^2} \tag{84}$$

and is shown below.

We also use that $F(-\omega) = F^*(\omega)$ if $F(t)$ is real. This means that

$$W = \frac{1}{2m} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \, |F(\omega)|^2 \, [\omega \mathrm{Im} G_R(\omega;\omega_+) + (\omega_+ \to \omega_-)] \tag{85}$$

We find finally

$$\chi(\omega) = \frac{1}{2m} [\omega \mathrm{Im} G_R(\omega;\omega_+) + (\omega_+ \to \omega_-)] \tag{86}$$

In the limit that the damping coefficient becomes zero, the result passes to part (c), since

$$\omega \mathrm{Im} G_R(\omega,\omega_0) \to \frac{2\pi}{4} [\delta(\omega-\omega_0) + \delta(\omega+\omega_0)] \tag{87}$$

as is clear from the figure above.

We show the result for $\chi(\omega)$ in Fig. 2

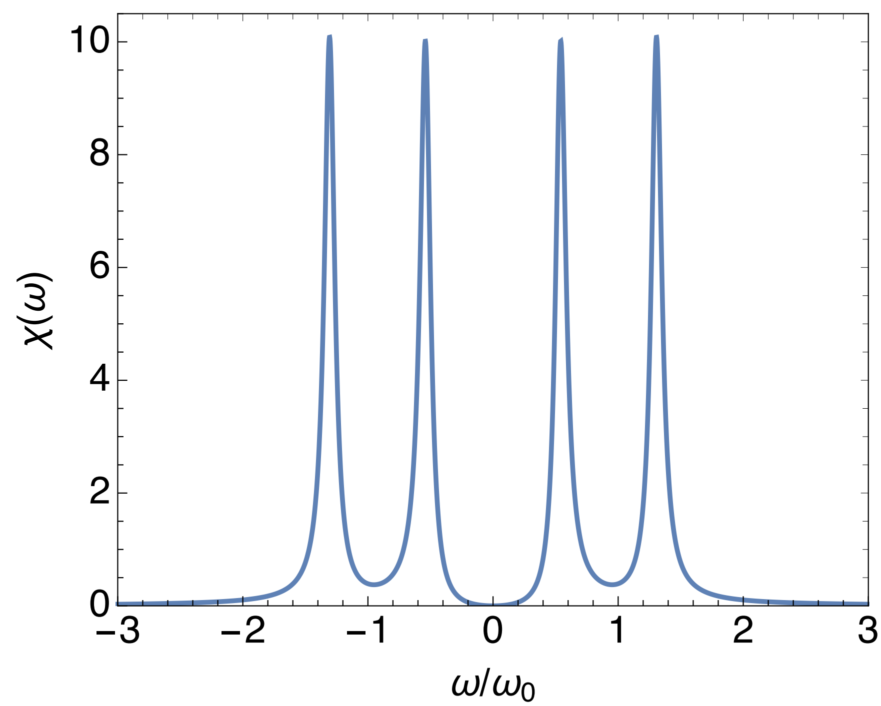


Figure 2: The four peaks are at $\omega = \pm\omega_+$ and $\omega = \pm\omega_-$