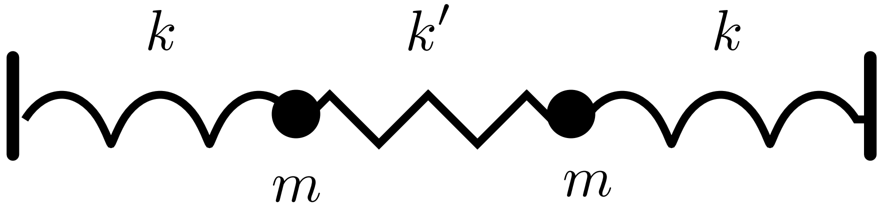
Problem 1. Oscillations with similar frequencies

Consider two particles of mass m coupled to the walls via springs with spring constant $k = m\omega_0^2$. The two particles are weakly coupled by a third spring with spring constant $k' = m\omega'^2$ as shown below. The particles can move only in the x-direction, and the springs are unstretched when the system is at rest. Assume that $\omega' \ll \omega_0$.



(a) If at time $t = 0$ the left particle is displaced by an initial position x_0 and the right particle is at rest, determine the subsequent oscillations of the system.

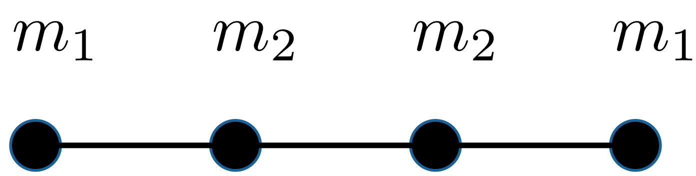
(b) Plot qualitatively $x_1(t)$ and $x_2(t)$ in regime where $k' \ll k$. Show all relevant features. Answer the following question: given a signal which is a sum of sinusoids

$$A\cos(\omega_1 t) + B\cos(\omega_2 t + \phi) \quad (1)$$

what is required to have prominent beats? Justify your answer with math.

Problem 2. Four masses with a kick

Consider the four masses depicted below which are connected by springs. The springs have spring constant k and the masses are $m_1 = m$ and $m_2 = 2m$. The masses move only in the x direction.



(a) Write down a set of coordinates which parameterize deformations that are even and odd and which are orthogonal to the zero modes.

(b) Write down the Lagrangian of the system in terms of the coordinates of part (a), and the center of mass coordinate. Determine the normal modes. You should only have to find the eigenvectors of one 2×2 matrix.

(c) If the left most mass is given an impulsive kick with force $F(t) = P_0\delta(t)$ at time $t = 0$, determine the positions of the particles at subsequent times. In what frame is the subsequent motion periodic?

Problem 3. (Goldstein) A molecule with a right triangle

The equilibrium configuration of a molecule consists of three identical atoms of mass m at the vertices of a 45^o right triangle connected by springs of equal force constant k. The atoms are constrained to move in the xy plane. We will determine the modes of oscillation of this molecule.

Zero Modes:

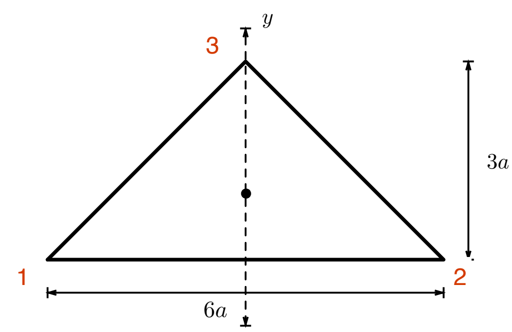
(a) The vectors in the space of displacements are labelled by

$$\vec{Q} = (x_1, y_1, x_2, y_2, x_3, y_3) \tag{2}$$

where (x_1, y_1) are the coordinates of particle 1, etc. Show that a displacement corresponding to a global rotation parameterized by the angle $\delta\theta$ around the z axis coming out of the page is

$$\vec{Q}_{\rm rot-z} = a\delta\theta\,(1, -3, 1, 3, -2, 0)\,. \tag{3}$$

Here we have chosen the long-length of the triangle to be $6a$ and the height of the triangle to be $3a$, the origin is taken to be the center of mass.



(b) Write down the other zero modes parametrized by the coordinates $X_{\rm cm}$ and $Y_{\rm cm}$.

Vibrational Modes:

(c) Under a reflection over the y axis the displacements $\vec{Q}$ are mapped to some new displacements $\underline{\vec{Q}}$. Explain qualitatively why $\underline{\vec{Q}}$

$$\vec{Q} \to \underline{\vec{Q}} = (\underline{x}_1, \underline{y}_1, \underline{x}_2, \underline{y}_2, \underline{x}_3, \underline{y}_3) = \underline{(} - x_2, y_2, -x_1, y_1, -x_3, y_3)\,.$$

We say that a vector is *odd* under reflection if $\underline{\vec{Q}} = -\vec{Q}$ and even under reflection if $\underline{\vec{Q}} = \vec{Q}$. Since the problem is symmetric under reflections, the eigenmodes will be either even or odd. The rotation in Eq. (3) is an eigenmode with zero eigenvalue. Is this mode even or odd?

(d) Show that there is only one *odd* basis vector (parameterized by a coordinate $q_{\rm o}(t)$) which is orthogonal to the three zero modes and determine its form. Then write down two (somewhat arbitrary) *even* basis vectors parameterized by two generalized coordinates $q_1(t)$ and $q_2(t)$ which are orthogonal to the zero modes, which you will use to parametrize the even oscillations.

(e) Write down the Lagrangian of the system using the six well chosen coordinates $(X_{\rm cm}, Y_{\rm cm}, \delta\theta, q_o, q_1, q_2)$ instead of $(x_1, y_1, x_2, y_2, x_3, y_3)$.

(f) Find the eigen-frequencies of the system and qualitatively sketch the non-zero vibrational modes. You should find

$$\omega^2 = \frac{3k}{m}, \frac{2k}{m}, \frac{k}{m} \tag{4}$$