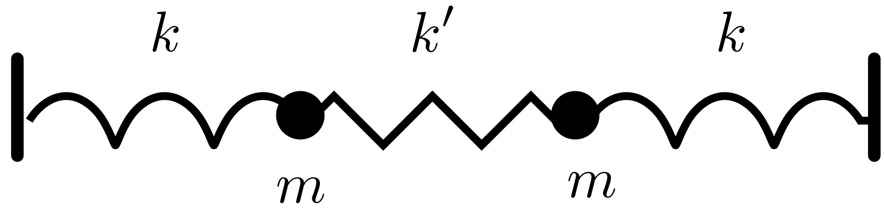
Problem 1. Oscillations with similar frequencies

Consider two particles of mass m coupled to the walls via springs with spring constant $k = m\omega_0^2$. The two particles are weakly coupled by a third spring with spring constant $k' = m\omega'^2$ as shown below. The particles can move only in the x-direction, and the springs are unstretched when the system is at rest. Assume that $\omega' \ll \omega_0$.



(a) If at time $t = 0$ the left particle is displaced by an initial position x_0 and the right particle is at rest, determine the subsequent oscillations of the system.

(b) Plot qualitatively $x_1(t)$ and $x_2(t)$ in regime where $k' \ll k$. Show all relevant features. Answer the following question: given a signal which is a sum of sinusoids

$$A\cos(\omega_1 t) + B\cos(\omega_2 t + \phi) \tag{1}$$

what is required to have prominent beats? Justify your answer with math.

Solution

The equations of motion are given by Newtons Laws.

(a) The equations of motion are

$$m\ddot{x}_1 = - kx_1 + k'(x_2 - x_1)\,, \tag{2}$$

$$m\ddot{x}_2 = - k'(x_2 - x_1) - kx_2\,. \tag{3}$$

The matrix reads

$$- m\omega^2 \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = - \begin{pmatrix} (k+k') & k' \\ k' & (k+k') \end{pmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}\,. \tag{4}$$

The matrix has the form $(k+k')\mathbb{I} + k'\sigma_x$ where σ_x is the pauli matrix. In the eigen-basis of σ_x we have

$$m\omega_\pm^2 = (k+k') \pm k'\,. \tag{5}$$

The eigenvectors of σ_x are

$$E_\pm = \begin{pmatrix} 1 \\ \mp 1 \end{pmatrix} \tag{6}$$

To a good approximation since $k \gg k'$ we have

$$\omega_\pm = \overline{\omega} \pm \Delta\omega\,, \tag{7}$$

where

$$\overline{\omega} = \frac{k+k'}{m}\,, \tag{8}$$

$$\Delta\omega = \frac{(\omega')^2}{2\overline{\omega}}\,. \tag{9}$$

The initial conditions a simple 50-50 superposition of E_+ and E_- leading to the solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{x_0}{2} e^{-i\omega_+ t} E_+ + \frac{x_0}{2} e^{-i\omega_- t} E_-\,. \tag{10}$$

(b) **Beats:** We have a superposition of sinusoids with nearly equal frequency and equal amplitude. This is a setup for very strong beats. So we need $A \simeq B$ and $\omega_1 \simeq \omega_2$.

Indeed we have taking the real part of

$$x_1 = x_0 \cos(\overline{\omega}t) \cos(\Delta\omega t)\,, \tag{11}$$

$$x_2 = - x_0 \sin(\overline{\omega}t) \sin(\Delta\omega t)\,. \tag{12}$$

Fig. 1 shows the solution for $\overline{\omega} = 30\Delta\omega$

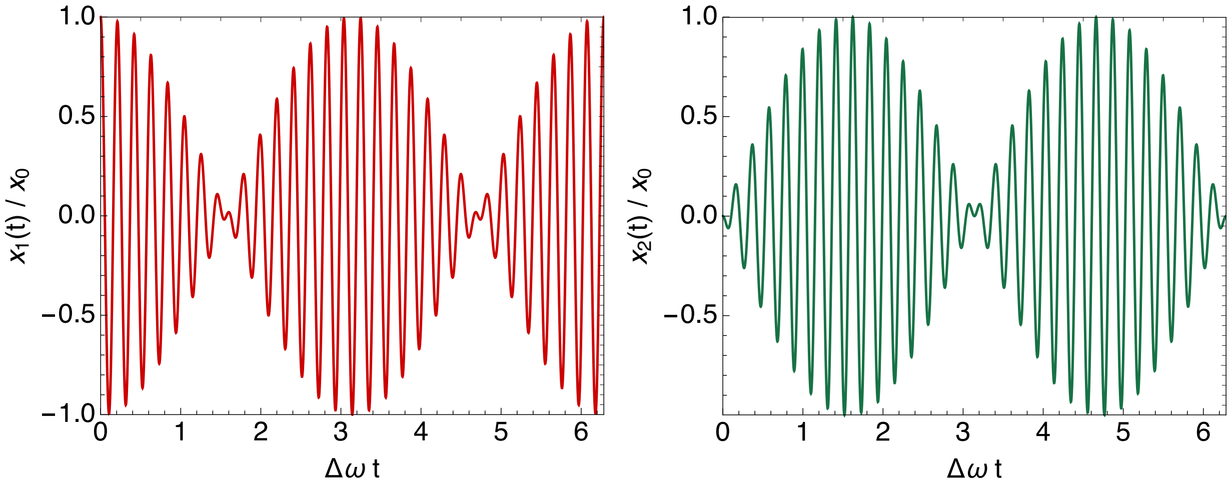
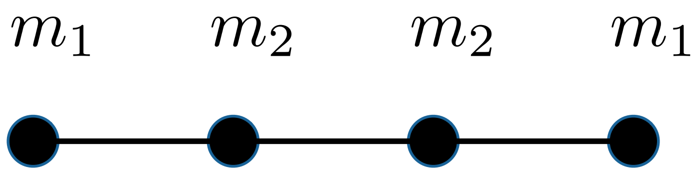


Figure 1: The solution for $\overline{\omega} = 30\Delta\omega$

Problem 2. Four masses with a kick

Consider the four masses depicted below which are connected by springs. The springs have spring constant k and the masses are $m_1 = m$ and $m_2 = 2m$. The masses move only in the x direction.



(a) Write down a set of coordinates which parameterize deformations that are even and odd and which are orthogonal to the zero modes.

(b) Write down the Lagrangian of the system in terms of the coordinates of part (a), and the center of mass coordinate. Determine the normal modes. You should only have to find the eigenvectors of one 2×2 matrix.

You mash wish to use Mathematica for making the substitution of (a). I found the frequencies to be

$$\omega^2 = \frac{2k}{m}, \quad \frac{k}{2m}, \quad \frac{3k}{2m}, \quad 0\,. \tag{13}$$

where the first two are the even modes, and the last two are the odds modes

(c) If the left most mass is given an impulsive kick with force $F(t) = P_0\delta(t)$ at time $t = 0$, determine the positions of the particles at subsequent times. In what frame is the subsequent motion periodic?

Solution

(a) We parametrize two even modes by coordinates

$$\vec{u} = q_1(1,0,0,-1) \tag{14}$$

and

$$\vec{u} = q_2(0,1,-1,0) \tag{15}$$

The zero mode is

$$E_0 = (1,1,1,1) \tag{16}$$

and the associated odd deformation is parametrized by the center of mass coordinate

$$\vec{u} = X\vec{E}_0 \tag{17}$$

Finally the only other odd mode which can be orthogonal to E_0 takes the form

$$\vec{u} = (x_1, x_2, x_3, x_4) = q_3(b,-1,-1,b) \tag{18}$$

Requiring it have no shift in the center of mass gives

$$bm - 2m - 2m + bm = 0 \qquad b = 2 \tag{19}$$

To summarize we use coordinates

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = q_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + q_2 \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + q_3 \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + X \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{20}$$

(b) Take the Lagrangian

$$L = \frac{1}{2} m_{ij} \dot{x}^i \dot{x}^j - U \tag{21}$$

where $m_{ij} = \mathrm{diag}(m, 2m, 2m, m)$ and

$$U = \frac{1}{2}k(x_2 - x_1)^2 + \frac{1}{2}k(x_3 - x_2)^2 + \frac{1}{2}k(x_4 - x_3)^2 \tag{22}$$

Making the substitutings

$$L = \frac{1}{2}\left(2m\dot{q}_1^2 + 4m\dot{q}_2^2\right) + \frac{1}{2}\,12m\,\dot{q}_3^2 + \frac{1}{2}M_{\rm tot}\dot{X}^2 - U \tag{23}$$

with

$$U = \frac{1}{2}\left(2k_1 q_1^2 - 4q_1 q_2 + 6q_2^2\right) + \frac{1}{2}18kq_3^2 \tag{24}$$

Reading off from the lagrangian we have immediately two normal modes parametrized by q_3 and X with frequencies.

$$\omega_0^2 = 0 \tag{25}$$

$$\omega_3^2 = \frac{18}{12}\frac{k}{m} = \frac{3}{2}\frac{k}{m} \tag{26}$$

The equations of q_1 and q_2 are coupled

$$m\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} = -k\begin{pmatrix} 2 & -2 \\ -2 & 6 \end{pmatrix}\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{27}$$

Solving the 2×2 evalue problem we have

$$\omega_1^2 = \frac{2k}{m} \tag{28}$$

$$\omega_2^2 = \frac{k}{2m} \tag{29}$$

with defformations

$$\begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = Q_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Q_2\begin{pmatrix} 2 \\ 1 \end{pmatrix} \tag{30}$$

In terms of the original coordinates we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = (Q_1 + 2Q_2)\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} + (-Q_1 + Q_2)\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} + q_3\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + X\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{31}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = Q_1\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + Q_2\begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix} + q_3\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + X\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{32}$$

$$\tag{33}$$

The associated frequencies are given by (in order)

$$\omega_1^2 = \frac{2k}{m}, \qquad \omega_2^2 = \frac{k}{2m}, \qquad \omega_3^2 = \frac{3}{2}\frac{k}{m}, \qquad \omega_0^2 = 0 \tag{34}$$

One can easily verify that

$$L = \frac{1}{2}\left[6m\dot{Q}_1^2 + 12m\dot{Q}_2^2 + 12m\dot{q}_3^2 + 6m\dot{X}^2\right] - \frac{1}{2}\left[12kQ_1^2 + 6kQ_2^2 + 18kq_3^2\right] \tag{35}$$

(c) The motion is periodic in the CM-frame, $v_{\rm cm} = \frac{P_0}{6m}$. In this fame particle 1 moves initially with velocity

$$v_{01} = 5v_{\rm cm} \tag{36}$$

while the others move with velocity

$$v_{02} = v_{03} = v_{04} = -v_{\rm cm} \tag{37}$$

So we have

$$X = v_{\rm cm} t \tag{38}$$

The general solution for a simple harmonic oscillator with initial displcement x_0 and initial velocity v_0 is

$$x = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) \tag{39}$$

We have only initial velocities, so our solution takes the form

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = V_{Q1} \frac{\sin(\omega_1 t)}{\omega_1} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + V_{Q2} \frac{\sin(\omega_3 t)}{\omega_3} \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix} + V_{Q3} \frac{\sin(\omega_3 t)}{\omega_3} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \tag{40}$$

The values of V_{Q1}, V_{Q2}, and V_{Q3} are given by the initial conditions

$$v_{\rm cm} \begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \end{pmatrix} = V_{Q1} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + V_{Q2} \begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix} + V_{Q3} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} \tag{41}$$

We can use the fact that the eigenvectors are orthogonal with metric

$$\mathcal{M} \equiv m \begin{pmatrix} 1 & & & \\ & 2 & & \\ & & 2 & \\ & & & 1 \end{pmatrix}. \tag{42}$$

Denoting

$$\vec{V}_0 = v_{\rm cm} \begin{pmatrix} 5 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \tag{43}$$

We have

$$V_{Q1} = \frac{\vec{V}_0 \cdot \vec{E}_1}{E_1 \cdot E_1} = v_{\rm cm} \tag{44}$$

$$V_{Q2} = \frac{\vec{V}_0 \cdot \vec{E}_2}{E_2 \cdot E_2} = v_{\rm cm} \tag{45}$$

$$V_{Q3} = \frac{\vec{V}_0 \cdot \vec{E}_3}{E_3 \cdot E_3} = v_{\rm cm} \tag{46}$$

$$\tag{47}$$

Putting together the ingrediants in the original frame

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = v_{cm}\frac{\sin(\omega_1 t)}{\omega_1}\begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix} + v_{cm}\frac{\sin(\omega_3 t)}{\omega_3}\begin{pmatrix} 2 \\ 1 \\ -1 \\ -2 \end{pmatrix} + v_{cm}\frac{\sin(\omega_3 t)}{\omega_3}\begin{pmatrix} 2 \\ -1 \\ -1 \\ 2 \end{pmatrix} + v_{cm}t\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{48}$$

Problem 3. (Goldstein) A molecule with a right triangle

The equilibrium configuration of a molecule consists of three identical atoms of mass m at the vertices of a 45^o right triangle connected by springs of equal force constant k. The atoms are constrained to move in the xy plane. We will determine the modes of oscillation of this molecule.

Zero Modes:

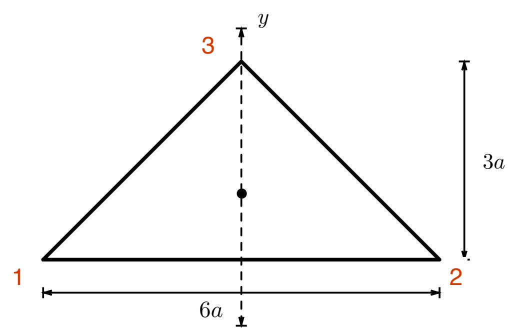
(a) The vectors in the space of displacements are labelled by

$$\vec{Q} = (x_1, y_1, x_2, y_2, x_3, y_3) \tag{49}$$

where (x_1, y_1) are the coordinates of particle 1, etc. Show that a displacement corresponding to a global rotation parameterized by the angle $\delta\theta$ around the z axis coming out of the page is

$$\vec{Q}_{\text{rot}-\text{z}} = a\delta\theta\,(1, -3, 1, 3, -2, 0)\,. \tag{50}$$

Here we have chosen the long-length of the triangle to be $6a$ and the height of the triangle to be $3a$, the origin is taken to be the center of mass.



(b) Write down the other zero modes parametrized by the coordinates X_{cm} and Y_{cm}.

Vibrational Modes:

(c) Under a reflection over the y axis the displacements $\vec{Q}$ are mapped to some new displacements $\underline{\vec{Q}}$. Explain qualitatively why $\underline{\vec{Q}}$

$$\vec{Q} \to \underline{\vec{Q}} = (\underline{x}_1, \underline{y}_1, \underline{x}_2, \underline{y}_2, \underline{x}_3, \underline{y}_3) = (-x_2, y_2, -x_1, y_1, -x_3, y_3)\,.$$

We say that a vector is *odd* under reflection if $\underline{\vec{Q}} = -\vec{Q}$ and even under reflection if $\underline{\vec{Q}} = \vec{Q}$. Since the problem is symmetric under reflections, the eigenmodes will be either even or odd. The rotation in Eq. 50 is an eigenmode with zero eigenvalue. Is this mode even or odd?

(d) Show that there is only one *odd* basis vector (parameterized by a coordinate $q_o(t)$) which is orthogonal to the three zero modes and determine its form. Then write down two (somewhat arbitrary) *even* basis vectors parameterized by two generalized coordinates $q_1(t)$ and $q_2(t)$ which are orthogonal to the zero modes, which you will use to parametrize the even oscillations.

(e) Write down the Lagrangian of the system using the six well chosen coordinates $(X_{\rm cm}, Y_{\rm cm}, \delta\theta, q_o, q_1, q_2)$ instead of $(x_1, y_1, x_2, y_2, x_3, y_3)$.

(f) Find the eigen-frequencies of the system and qualitatively sketch the non-zero vibrational modes. You should find

$$\omega^2 = \frac{3k}{m}, \frac{2k}{m}, \frac{k}{m} \tag{51}$$

Solution:

(a) There are two translational zero modes and one rotational zero mode. If the vector space of displacements is denoted

$$\vec{Q} = (x_1, y_1, x_2, y_2, x_3, y_3) \tag{52}$$

Then the translational zero modes in the x and y directions are

$$\vec{T}_x = (1, 0, 1, 0, 1, 0) \tag{53}$$

$$\vec{T}_y = (0, 1, 0, 1, 0, 1) \tag{54}$$

To work out the rotational zero modes we need to set up a coordinate system Placing the center of mass at the origin, the coordinates of the atoms are

$$\boldsymbol{r}_{10}, \boldsymbol{r}_{20}, \boldsymbol{r}_{30} \tag{55}$$

If we call the long length $6a$ then

$$\boldsymbol{r}_{10} = (-3a, -a) \equiv \boldsymbol{n}_1 \tag{56}$$

$$\boldsymbol{r}_{20} = (3a, -a) \equiv \boldsymbol{n}_2 \tag{57}$$

$$\boldsymbol{r}_{30} = (0, 2a) \equiv \boldsymbol{n}_3 \tag{58}$$

The disturbances drawn in the figure are orthogonal to these vectors. Thus the vector displacement associated with the first atom is

$$\boldsymbol{z} \times \boldsymbol{r}_{10} = (a, -3a) \equiv \boldsymbol{m}_1 \tag{59}$$

and similarly

$$\boldsymbol{z} \times \boldsymbol{r}_{10} = (a, -3a) \equiv \boldsymbol{m}_1 \tag{60}$$

$$\boldsymbol{z} \times \boldsymbol{r}_{20} = (a, 3a) \equiv \boldsymbol{m}_2 \tag{61}$$

$$\boldsymbol{z} \times \boldsymbol{r}_{30} = (-2a, 0) \equiv \boldsymbol{m}_3 \tag{62}$$

Thus the rotational zero mode is

$$\vec{R}_z = a(1, -3, 1, 3, -2, 0) = (\boldsymbol{m}_1, \boldsymbol{m}_2, \boldsymbol{m}_3) \tag{63}$$

(b) Now we perturb the problem But in general we consider $\vec{Q}$ which lies in a subspace orthogonal to the zero modes. We can also divide $\vec{Q}$ into directions which are even / odd under the reflection symmetries. So out the six numbers in $\vec{Q}$ only three are independent. One could choose these to be x_1, y_1, x_2 in a pragmatic fashion. However we are motivated by symmetry, and choose as our three independent coordinates

$$x_\pm = (x_1 \pm x_2)/2 \qquad y_3 \tag{64}$$

So

$$((x_+ + x_-, y_1), (x_+ - x_-, y_2),\ (x_3, y_3)) \tag{65}$$

Requiring that this is orthogonal to the $T_{x,y}$ and R_z yields a parameterization of a vector which does not shit the center of mass or cause a rotation

$$\vec{Q} = ((x_+ + x_-, x_+ - \frac{1}{2}y_3),\ (x_+ - x_-, -x_+ - \frac{1}{2}y_3),\ (-2x_+, y_3)) \tag{66}$$

A figure below shows the how the three displacements distort the molecule. We label this displacements $\vec{E}_+$ and $\vec{E}_x$ and $\vec{E}_y$

$$\vec{Q} = x_+ \underbrace{(1,1,1,-1,-2,0)}_{\vec{E}_+} + x_- \underbrace{(1,0,-1,0,0,0)}_{\vec{E}_x} + y_3 \underbrace{(0,-\frac{1}{2},0,-\frac{1}{2},0,1)}_{\vec{E}_y} \tag{67}$$

From the symmetry of the problem the $\vec{E}_x$ and $\vec{E}_y$ displacements can mix with each other (they are both even under the reflection symmetry). But because of the symmetry of the problem the $\vec{E}_+$ (which is odd under the reflection symmetry) can not mix with $\vec{E}_x$ and $\vec{E}_y$, and therefore must be an eigenvector. We will verify this below.

The potential energy

$$U = \frac{1}{2}k(\ell_{12} - \ell_{12}^o)^2 + \frac{1}{2}k(\ell_{31} - \ell_{31}^o)^2 + \frac{1}{2}k(\ell_{32} - \ell_{32}^o)^2 \tag{68}$$

where for example

$$\ell_{ab} = \sqrt{(\boldsymbol{r}_{0a} + \boldsymbol{r}_a - \boldsymbol{r}_{0b} - \boldsymbol{r}_b)^2} \tag{69}$$

and for example

$$\ell_{12}^o = \sqrt{(\boldsymbol{r}_{01} - \boldsymbol{r}_{02})^2} = 6a \tag{70}$$

Straightforward computer algebra gives

$$U = \frac{1}{2}\begin{pmatrix} x_+ & x_- & y_3 \end{pmatrix} \underbrace{k\begin{pmatrix} 16 & 0 & 0 \\ 0 & 5 & -3/2 \\ 0 & -3/2 & 9/4 \end{pmatrix}}_{\equiv K} \begin{pmatrix} x_+ \\ x_- \\ y_3 \end{pmatrix} \tag{71}$$

The potential would not have been so simple if we did not use $x_\pm$ and y_3. Then the kinetic energy in this basis is

$$\frac{1}{2}m\dot{Q}\cdot\dot{Q} = \frac{1}{2}\begin{pmatrix} \dot{x}_+ & \dot{x}_- & \dot{y}_3 \end{pmatrix} \underbrace{m\begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3/2 \end{pmatrix}}_{\equiv M} \begin{pmatrix} \dot{x}_+ \\ \dot{x}_- \\ \dot{y}_3 \end{pmatrix} \tag{72}$$

Then we can find the eigen frequencies through a straightforward diagonalization

$$\det(K - \omega^2 M) = 0 \tag{73}$$

This yields

$$\omega^2 = \frac{3k}{m}, \frac{2k}{m}, \frac{k}{m} \tag{74}$$

As anticipated, one of the eigen mode only involves x_+

$$\vec{E}_+ = E_{2k/m} = (1, 1, 1, -1, -2, 0) \tag{75}$$

The remaining eigenvectors are superpositions of $\vec{E}_x$ and $\vec{E}_y$.