Problem 1. A particle in a magnetic field

(a) Write down the Lagrangian and Hamiltonian for a particle in a magnetic field, $\boldsymbol{B}(\boldsymbol{r})$. Compute the Poisson brackets of velocity:

$$\{v_i, v_j\}$$

(b) Prove that the value of any function $f(q(t), p(t))$ of coordinates and momenta of a system at a time t can be expressed in terms of the values of p and q at $t = 0$ as follows:

$$f = f_0 + \frac{t}{1!}\{f_0, H\} + \frac{t^2}{2!}\{\{f_0, H\}, H\} + \dots, \tag{1}$$

where $f_0 = f(p(0), q(0))$. Apply this formula to evaluate $p^2(t)$ for a harmonic oscillator.

(c) Evaluate $\boldsymbol{v}(t)$ for a particle in a constant magnetic field $\boldsymbol{B}_0$ using the results of this problem.

Problem 2. Canonical transformations and Poisson Brackets

Consider an infinitesimal change of coordinates, which is not necessarily canonical:

$$q \to Q = q + \lambda \frac{dQ(q,p)}{d\lambda}\,, \tag{2}$$

$$p \to P = p + \lambda \frac{dP(q,p)}{d\lambda}\,. \tag{3}$$

Show that if the Poisson bracket is to remain fixed under the transformation, i.e.

$$\{Q, P\} = 1\,, \tag{4}$$

$$\{P, P\} = 0\,, \tag{5}$$

$$\{Q, Q\} = 0\,, \tag{6}$$

then there must exist a $G(q, p)$ which generates the transformation. (Hint recall the following theorem: if a vector field is curl free, $\nabla \times \boldsymbol{v} = 0$ it may be written as a gradient of a scalar function, $\boldsymbol{v} = -\nabla\phi$.)

Problem 3. 2d isotropic oscillator

Consider the 2d harmonic oscillator which is isotropic

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + (\omega_0 x_1)^2 + (\omega_0 x_2)^2\right) \tag{7}$$

This is an example of an integrable system, which means if the phase space consists of $2n$ generalized coordinates there are $2n-1$ constants of the motion. We will find and interpret these constants here.

(a) Show that

$$J_3(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2}\left(x_1 p_2 - p_1 x_2\right) \tag{8}$$

generates rotations in the plane. Why is it constant in time?

(b) Determine the infinitesimal transformation generated by

$$J_1(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2\omega_0}\left(\frac{1}{2}p_1^2 + \frac{1}{2}\omega_0^2 x_1^2 - \frac{p_2^2}{2} - \frac{1}{2}\omega_0^2 x_2^2\right), \tag{9}$$

and describe this transformation qualitatively[1]. Show that the computed transformation leaves the Hamiltonian invariant, and that this implies that $\dot{J}_1 = \{J_1, H\} = 0$. Give a physical interpretation of J_1.

(c) Use the Poisson theorem to deduce a third conserved quantity J_2:

$$J_2 = \frac{1}{2\omega_0}\left(p_1 p_2 + \omega_0^2 x_1 x_2\right) \tag{10}$$

Determine the associated infinitesimal canonical transformation generated by this conservation law, and verify that it is a symmetry of the Hamiltonian.

(d) We have found three integrals of motion. Using similar manipulations to part (c), one may show that

$$\{J_i, J_j\} = i\epsilon_{ijk} J_k\,, \tag{11}$$

and that

$$\left(\frac{H}{2\omega_0}\right)^2 = J_1^2 + J_2^2 + J_3^2 \tag{12}$$

Thus any random orbit is selected by choosing J_1, J_2, J_3 to lie on the surface of a sphere. Describe the motion of the orbit in each of the following limiting cases

(i) $J_1 = J_2 = 0$

(ii) $J_2 = J_3 = 0$

(iii) $J_1 = J_3 = 0$

[1] For example in part (a) we qualitatively said that J_3 generates rotations in the plane. Give a similar qualitative description for J_1.

(e) (Optional:) Consider the $2D$ oscillator in cylindrical coordinates

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}\omega_0^2 r^2 \tag{13}$$

Consider a particle in this potential is going around in a circle. At $t = 0$ it is on the x axis, and is then given a small extra push of impulse Δp in the y direction. Using the integrals of motion explain (without detailed calculation) why the perturbed orbit remains closed.