## Problem 1. A particle in a magnetic field

(a) Write down the Lagrangian and Hamiltonian for a particle in a magnetic field, $\boldsymbol{B}(\boldsymbol{r})$. Compute the Poisson brackets of velocity:

$$\{v_i, v_j\}$$

(b) Prove that the value of any function $f(q(t), p(t))$ of coordinates and momenta of a system at a time $t$ can be expressed in terms of the values of $p$ and $q$ at $t = 0$ as follows:

$$f = f_0 + \frac{t}{1!}\{f_0, H\} + \frac{t^2}{2!}\{\{f_0, H\}, H\} + \dots , \tag{1}$$

where $f_0 = f(p(0), q(0))$. Apply this formula to evaluate $p^2(t)$ for a harmonic oscillator.

(c) Evaluate $\boldsymbol{v}(t)$ for a particle in a constant magnetic field $\boldsymbol{B}_0$ using the results of this problem.

(a) For $\boldsymbol{B}(\boldsymbol{r})$, the action of the particle in an electromagnetic field is

$$S = \int dt \, \frac{1}{2} m \dot{\boldsymbol{v}}^2 + e \frac{\boldsymbol{v}}{c} \cdot \boldsymbol{A} \tag{2}$$

Then computing the momentum

$$\boldsymbol{p} = \frac{\partial L}{\partial(\partial_t \boldsymbol{v})} = m\boldsymbol{v} + \frac{e}{c}\boldsymbol{A}(\boldsymbol{r}) \tag{3}$$

So the Poisson brackets of velocity read

$$\{v_i, v_j\} = \frac{1}{m^2}\{p_i - \frac{e}{c}A_i(\boldsymbol{r}), p_j - \frac{e}{c}A_j(\boldsymbol{r})\} \tag{4}$$

Using the fact that

$$\{p_i, p_j\} = 0 \tag{5}$$
$$\{p_i, x_j\} = -\delta_{ij} \tag{6}$$

we can deduce that for any function $f(\boldsymbol{r})$

$$\{p_i, f(\boldsymbol{r})\} = -\frac{\partial f}{\partial x^i} \tag{7}$$

You can prove this by induction for $x^n$. Then any analytic function (which has a Taylor series) is also proved.

$$\{v_i, v_j\} = \frac{e}{m^2 c}\left(\frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x^j}\right) = \frac{e}{m^2 c}\epsilon_{ijk}B^k \tag{8}$$

where we used that $\boldsymbol{B} = \nabla \times \boldsymbol{A}$.

(b) One simply uses a Taylor series expansion of $f(p(t), q(t))$ as a function of $t$. The first few terms are (at any time)

$$\dot{f}(p,q) = \{f(p,q), H\} \tag{9}$$
$$\ddot{f}(p,q) = \{\dot{f}(p,q), H\} = \{\{f, H\}, H\} \tag{10}$$
$$\dddot{f}(p,q) = \{\ddot{f}(p,q), H\} = \{\{\{f, H\}, H\}, H\} \tag{11}$$

(c) The Hamiltonian is

$$H = \frac{1}{2} m v(\boldsymbol{p}, \boldsymbol{r})^2 \tag{12}$$

Then evaluating the Poisson bracket

$$\frac{dv_j}{dt} = \{v_j, H\} = \frac{1}{mc}\epsilon_{jik}v^i B^k \tag{13}$$

yields the right equation of motion. Taking $B_0$ along the $z$ direction Our commutators read

$$\{v_x, H\} = \frac{eB_0}{mc} v^y \tag{14}$$

$$\{v_y, H\} = -\frac{eB_0}{mc} v^x \tag{15}$$

$$\{v_z, H\} = 0 \tag{16}$$

Then higher commutators read

$$\{\{v_x, H\}, H\} = -\left(\frac{eB_0}{mc}\right)^2 v^x \tag{17}$$

$$\{\{v_y, H\}, H\} = -\left(\frac{eB_0}{mc}\right)^2 v^y \tag{18}$$

and still higher read

$$\{\{\{v_x, H\}, H\}, H\} = -\left(\frac{eB_0}{mc}\right)^3 v^y \tag{19}$$

$$\{\{\{v_y, H\}, H\}, H\} = +\left(\frac{eB_0}{mc}\right)^3 v^y \tag{20}$$

etc

So we find, using the result of the previous problem, that first of all

$$v_z(t) = v_z(0)\,. \tag{21}$$

Then for $v_x$ and $v_y$ we have after collecting terms proportional to $v_x(0)$ and $v_y(0)$ :

$$v_x(t) = v_x(0)\left[1 - \frac{t^2}{2!}\left(\frac{eB_0}{mc}\right)^2 + \dots\right] + v_y(0)\left[\frac{eB_0}{mc} - \frac{t^3}{3!}\left(\frac{eB_0}{mc}\right)^3 + \dots\right] \tag{22}$$

$$v_y(t) = v_y(0)\left[1 - \frac{t^2}{2!}\left(\frac{eB_0}{mc}\right)^2 + \dots\right] - v_x(0)\left[\frac{eB_0}{mc} - \frac{t^3}{3!}\left(\frac{eB_0}{mc}\right)^3 + \dots\right] \tag{23}$$

i.e.

$$v_x(t) = v_x(0)\cos(eB_0/mct) + v_y(0)\sin(eB_0/mct) \tag{24}$$

$$v_y(t) = -v_x(0)\sin(eB_0/mct) + v_y(0)\cos(eB_0/mct) \tag{25}$$

Of course

$$v_x(t)^2 + v_y(t)^2 = v_x(0)^2 + v_y(0)^2 \tag{26}$$

indicating that the particle is going in a circle.

## Problem 2. Canonical transformations and Poisson Brackets

Consider an infinitesimal change of coordinates, which is not necessarily canonical:

$$q \to Q = q + \lambda \frac{dQ(q,p)}{d\lambda}\,, \tag{27}$$

$$p \to P = p + \lambda \frac{dP(q,p)}{d\lambda}\,. \tag{28}$$

Show that if the Poisson bracket is to remain fixed under the transformation, i.e.

$$\{Q,P\} = 1\,, \tag{29}$$

$$\{P,P\} = 0\,, \tag{30}$$

$$\{Q,Q\} = 0\,, \tag{31}$$

then there must exist a $G(q,p)$ which generates the transformation. (Hint recall the following theorem: if a vector field is curl free, $\nabla \times \boldsymbol{v} = 0$ it may be written as a gradient of a scalar function, $\boldsymbol{v} = -\nabla\phi$.)

**Solution:**

Substituting, we require

$$\{q + \lambda\Delta q, p + \Delta p\} = 1 \tag{32}$$

leading to

$$\lambda\{\Delta q, p\} + \lambda\{q, \Delta p\} = 0 \tag{33}$$

With the definition of the Poisson bracket

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial B}{\partial q}\frac{\partial A}{\partial p} \tag{34}$$

We find

$$\lambda\left(\frac{\partial \Delta q}{\partial q} + \frac{\partial \Delta p}{\partial p}\right) = 0\,. \tag{35}$$

It is helpful here to use the $2D$ notation from class writing

$$\boldsymbol{v} = (-\Delta p, \Delta q)\,. \tag{36}$$

The vector $\boldsymbol{v}$ is the the symplectic matrix times $(\Delta q, \Delta p)$

$$\begin{pmatrix} -\Delta p \\ \Delta q \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \Delta q \\ \Delta p \end{pmatrix} \tag{37}$$

Thus the condition $\{Q, P\} = 1$ is written

$$\partial_x v_y - \partial_y v_x = 0\,. \tag{38}$$

Given this curl free result we we can write $\boldsymbol{v}$ as the gradient of a scalar

$$\boldsymbol{v} = \nabla G = (\partial_q G, \partial_p G) = (-\Delta p, \Delta q)\,. \tag{39}$$

Thus the transformation rule is

$$q \to Q = q + \lambda\partial_p G \tag{40}$$

$$p \to P = p - \lambda\partial_q G \tag{41}$$

## Problem 3. 2d isotropic oscillator

Consider the 2d harmonic oscillator which is isotropic

$$H = \frac{1}{2}\left(p_1^2 + p_2^2 + (\omega_0 x_1)^2 + (\omega_0 x_2)^2\right) \tag{42}$$

This is an example of an integrable system, which means if the phase space consists of $2n$ generalized coordinates there are $2n - 1$ constants of the motion. We will find and interpret these constants here.

(a) Show that

$$J_3(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2}\left(x_1 p_2 - p_1 x_2\right) \tag{43}$$

generates rotations in the plane. Why is it constant in time?

(b) Determine the infinitesimal transformation generated by

$$J_1(\boldsymbol{r}, \boldsymbol{p}) = \frac{1}{2\omega_0}\left(\frac{1}{2}p_1^2 + \frac{1}{2}\omega_0^2 x_1^2 - \frac{p_2^2}{2} - \frac{1}{2}\omega_0^2 x_2^2\right), \tag{44}$$

and describe this transformation qualitatively[1]. Show that the computed transformation leaves the Hamiltonian invariant, and that this implies that $\dot{J}_1 = \{J_1, H\} = 0$. Give a physical interpretation of $J_1$.

(c) Use the Poisson theorem to deduce a third conserved quantity $J_2$:

$$J_2 = \frac{1}{2\omega_0}\left(p_1 p_2 + \omega_0^2 x_1 x_2\right) \tag{45}$$

Determine the associated infinitesimal canonical transformation generated by this conservation law, and verify that it is a symmetry of the Hamiltonian.

(d) We have found three integrals of motion. Using similar manipulations to part $(c)$, one may show that

$$\{J_i, J_j\} = i\epsilon_{ijk} J_k \,, \tag{46}$$

and that

$$\left(\frac{H}{2\omega_0}\right)^2 = J_1^2 + J_2^2 + J_3^2 \tag{47}$$

Thus any random orbit is selected by choosing $J_1, J_2, J_3$ to lie on the surface of a sphere. Describe the motion of the orbit in each of the following limiting cases

(i) $J_1 = J_2 = 0$

(ii) $J_2 = J_3 = 0$

(iii) $J_1 = J_3 = 0$

(e) (Optional:) Consider the $2D$ oscillator in cylindrical coordinates

$$L = \frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{1}{2}\omega_0^2 r^2 \tag{48}$$

Consider a particle in this potential is going around in a circle. At $t = 0$ it is on the $x$ axis, and is then given a small extra push of impulse $\Delta p$ in the $y$ direction. Using the integrals of motion explain (without detailed calculation) why the perturbed orbit remains closed.

[1] For example in part (a) we qualitatively said that $J_3$ generates rotations in the plane. Give a similar qualitative description for $J_1$.

(a). Under infinitesimal transformation

$$x_1 \to x_1 + \lambda \frac{\partial J_3}{\partial p_1} \tag{49}$$

$$x_2 \to x_2 + \lambda \frac{\partial J_3}{\partial p_2} \tag{50}$$

which is is rotation

$$x_1 \to x_1' = x_1 - \lambda x_2 \tag{51}$$

$$x_2 \to x_2' = x_2 + \lambda x_1 \tag{52}$$

This a rotation in clockwise fashion of the coordinates $(x_1, x_2)$ by an amount $\delta\theta = \lambda$. It also generates a similar rule for $(p_1, p_2)$

$$p_1 \to p_1' = p_1 - \lambda p_2 \tag{53}$$

$$p_2 \to p_2' = p_2 + \lambda p_1 \tag{54}$$

$J_3$ is constant in time because this transformation rule leaves $H$ unchanged. $H$ is unchanged because it is a rotation which leaves the norm of vectors fixed,

$$p_a p_a = p_a' p_a', \qquad x_a x_a = x_a' x_a' \,. \tag{55}$$

(b) The generator $J_1$ yields the following transformation for $x_1, p_1$

$$\omega_0 x_1 \to \omega_0 x_1' = \omega_0 x_1 + \lambda \frac{\partial J_1}{\partial p_1} \tag{56}$$

$$= \omega_0 x_1 + \frac{\lambda}{2} p_1 \tag{57}$$

$$p_1 \to p_1' = p_1 - \lambda \frac{\partial J_1}{\partial p_1} \tag{58}$$

$$= p_1 - \frac{\lambda}{2} \omega_0 x_1 \tag{59}$$

and an analogous relation for $x_2, p_2$

$$\omega_0 x_2 \to \omega_0 x_2' = \omega_0 x_2 + \lambda \frac{\partial J_1}{\partial p_2} \tag{60}$$

$$= \omega_0 x_2 - \frac{\lambda}{2} p_2 \tag{61}$$

$$p_2 \to p_2' = p_2 - \lambda \frac{\partial J_1}{\partial p_2} \tag{62}$$

$$= p_2 + \frac{\lambda}{2} \omega_0 x_2 \tag{63}$$

This is a rotation in of the "vectors" $(\omega_0 x_1, p_1)$ and $(\omega_0 x_2, p_2)$ by an angle proportional to $\lambda$ and $-\lambda$ respectively. The Hamiltonian involves the sum of squares of these vectors

$$H \propto (p_1^2 + (\omega_0 x_1)^2) + ((p_2)^2 + (\omega_0 x_2)^2) \tag{64}$$

Each square is invariant under such rotations leaving $H$ unchanged, and therefore the generator of this transformation $J_1$ is conserved. $J_1$ is the difference in energy between the $x$-vibrations and the $y$-vibrations.

(c) The Poisson theorem says that if $J_1$ is conserved and $J_3$ then so is $\{J_3, J_1\}$. The proof is straightforward using the Jacobi identity:

$$\{J_1, \{J_3, H\} + \{J_3, \{H, J_1\}\} + \{H, \{J_1, J_3\}\} = 0 \tag{65}$$

Since $J_1$ and and $J_3$ both commute the the Hamiltonian, i.e. $\{J_1, H\} = 0$, we see that $\{J_1, J_3\}$ also commutes with the Hamiltonian, i.e. it is conserved locally.

Computing the necessary Poisson bracket

$$\begin{aligned}
\{J_3, J_1\} =& \{\frac{1}{2}(x_1p_2 - p_1x_2), \frac{1}{2\omega_0}\left(\frac{1}{2}p_1^2 + \frac{1}{2}\omega_0^2x_1^2 - \frac{p_2^2}{2} - \frac{1}{2}\omega_0^2x_2^2\right)\} && (66)\\
=& \frac{1}{4\omega_0}\{x_1p_2, \tfrac{1}{2}(p_1^2 - \omega_0^2x_2^2)\} - \frac{1}{4\omega_0}\{p_1x_2, \tfrac{1}{2}(\omega_0^2x_1^2 - p_2^2)\} && (67)\\
=& \frac{1}{4\omega_0}\left(p_1p_2 + \omega_0^2x_1x_2\right) - \frac{1}{4\omega_0}(-\omega_0^2x_2x_1 - p_1p_2) && (68)\\
=& \frac{1}{2\omega_0}(p_1p_2 + \omega_0^2x_1x_2) && (69)
\end{aligned}$$

The transformation generated by this generator is

$$\begin{aligned}
x_1 \to& x_1' = x_1 + \frac{\lambda}{2}\frac{p_2}{\omega_0} && (70)\\
p_1 \to& p_1' = p_1 - \frac{\lambda}{2}\omega_0x_2 && (71)\\
x_2 \to& x_2' = x_2 + \frac{\lambda}{2}\frac{p_1}{\omega_0} && (72)\\
p_2 \to& p_2' = p_2 - \frac{\lambda}{2}\omega_0x_1 && (73)
\end{aligned}$$

This is a rotation of the "vectors" $(\omega_0x_1, p_2)$ and $(\omega_0x_2, p_2)$ by $\lambda/2$. The Hamiltonian is sum of the squares of these vectors

$$H \propto (\omega_0^2x_1^2 + p_2^2) + (\omega_0^2x_2^2 + p_1^2) \tag{74}$$

and is invariant under these rotations, since the lengths of these "vectors" are unchanged by the rotation. $J_2$ is related by a rotation to $J_1$. Indeed of we rotate our coordinate system by $\pi/4$, $J_1$ becomes $J_2$ and $J_2$ becomes $J_1$. This is easy to see. For a $\pi/4$ rotation

$$\begin{aligned}
x_1' =& \frac{x_1 + x_2}{\sqrt{2}} && (75)\\
x_2' =& \frac{x_1 - x_2}{\sqrt{2}} && (76)
\end{aligned}$$

and thus

$$x'_1 x'_2 = \frac{1}{2}(x_1^2 - x_2^2) \tag{77}$$

$J_2$ is thus the difference in energy between the vibrations along the two diagonals.

(d) In the first case $J_1 = J_2 = 0$. The energy in the $x, y$ direction is the same. Looking at $J_1$ this means that when the oscillator in the $x_1$ direction has maximal potential energy ($p_1 = 0$, $\omega_0 x_1$ max), the oscillator in the $y$ direction has maximal kinetic energy ($p_2$ max, $\omega_0 x_2 = 0$) and zero potential energy. The two oscillators in the $x, y$ direction thus have equal amplitude and are out of phase by $\pi/2$. Thus the motion is circular and this is a circular orbit.

In the second case $J_2 = J_3 = 0$. The oscillator motion has no angular momentum. The "orbital" motion is thus along a line. Since $J_2$ is zero. We may set $p_2 = x_2 = 0$ (case1) or $p_1 = x_1 = 0$ (case2). Then if $J_1 > 0$ (case 1), $J_1$ is simply the energy of the $x_1$ motion of oscillations along the $x$ axis. If $J_1 < 0$ then all of the linear motion is along the $y$ axis.

In the final case $J_1 = J_3 = 0$ the energy of the $x$ and $y$ oscillators are equal. There is no angular momentum $J_3$ and thus the motion is linear. Since the energy of the two oscillators ($x_1$ direction and $x_2$ direction) the the motion is along the diagonal. If $J_2 > 0$ then the motion picks one of the diagonals, and if $J_2 < 0$ it picks the other. Examining the sign, we see that at maximal displacement in the first quadrant, $x_1 = x_2 = A$, and $p_1 = p_2 = 0$, and $J_2$ is therefore positive. Thus if $J_2 > 0$ the motion is along the diagonal in the first and third quadrant. If $J_2 < 0$ it is along the diagonal in the second and fourth.