Problem 1. Equations of motion

(a) From the Euler-Lagrange equations, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - V(q)$$

where $V(q) = \lambda q^4$.

(b) By varying the action, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha q \partial_x^4 q$$

(c) By varying the action, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha(\partial_x^2 q)^2$$

Compare to part (b) and comment on similarities and differences.

Problem 2. Group velocity

(a) (Optional – Warm up) This is lecture material, you can just copy the derivation, or skip it. Find the eigen-frequencies and normal modes of N particles which are connected by springs and which can move along a circle, *i.e* periodic boundary conditions $q_{N/2}(t) = q_{-N/2}(t)$. All particles have mass m and are separated by a distance a; the spring constants of all of the springs are the same and equal γ.

Express a general real solution $q_\ell(t)$ with $\ell = 1 \dots N$ as a linear superposition of the eigen-modes.

(b) Explain why the work done per time by the $j-1$th mass on the j-th mass is

$$\frac{dW}{dt} = -\gamma \dot{q}_j q_j + \gamma \dot{q}_j q_{j-1}$$

(c) Show that if the motion is that of a wave traveling along the circle $q_j = \mathcal{A} e^{ikx - i\omega(k)t}$, the time averaged energy flux (i.e. the time average of the work in (b)) equals the product of the (time averaged) energy per site and the group velocity of the system.

Here are some intermediate steps:

(i) (Optional, but recommended.) For computing the averages, prove the following result: if $A(t) = \mathrm{Re}[A_\omega e^{-i\omega t}]$ and $B(t) = \mathrm{Re}[B_\omega e^{-i\omega t}]$ then

$$\overline{A(t)B(t)} = \frac{1}{4}\left(A_\omega B_\omega^* + A_\omega^* B_\omega\right) = \frac{1}{2}\mathrm{Re}[A_\omega B_\omega^*] \tag{1}$$

Prove this result for yourself, by writing $A(t)$ as $\frac{1}{2}(A_\omega e^{-i\omega t} + A_\omega^* e^{+i\omega t})$.

We will use this below where $A_\omega = \mathcal{A} e^{ikx}$ or something similar.

(ii) Show that the time averaged kinetic energy per site and the time averaged potential energy per site are

$$\overline{T} = \frac{1}{4} m\omega^2(k)|\mathcal{A}|^2 = \frac{1}{4}\gamma|\mathcal{A}|^2(2\sin(ka/2))^2 \tag{2}$$

$$\overline{U} = \frac{1}{4}\gamma|\mathcal{A}|^2(2\sin(ka/2))^2 \tag{3}$$

(iii) Complete task (c) described above.

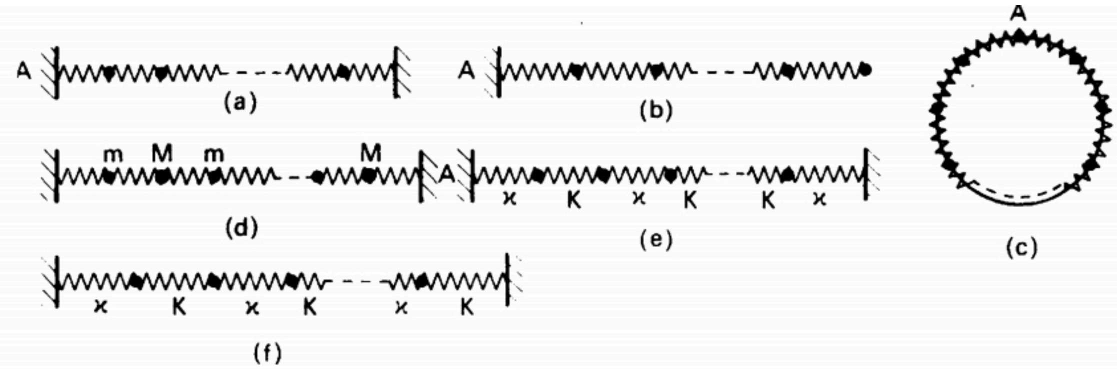

FIG. 33

Figure 1: A number of figures for the problems on springs

Problem 3. Normal modes and standing waves

In class we considered periodic boundary conditions. Here you will work out the modifications to the eigenfrequencies and functions with different boundary conditions.

(a) Determine the frequencies and eigenmodes of the eigen-vibrations for a system of N identical particles with masses m connected by identical springs with elastic constants γ and spatial separation a. Take the end points of the chain to be fixed (Fig. 33a).

You should find that general solution is a superposition of normal modes $\propto \sin(kx)$ labelled by an integer m

$$q_j(t) = \sum_{m=1}^{N} \mathcal{A}_m \cos(-\omega(k_m)t + \varphi_m) \sin(k_m j a) \tag{4}$$

with

$$k_m = \frac{\pi}{a}\frac{m}{N+1} \qquad\qquad m = 1 \ldots N \tag{5}$$

where $\mathcal{A}_m$ and φ_m are determined by the initial conditions $q_j(0)$ and $\dot{q}_j(0)$.

(b) (Optional) Repeat (a) when only one end is fixed, and the other end may vibrate freely (Fig. 33b).

You should find that the general solution is a superposition of normal modes

$$q_j(t) = \sum_{m=0}^{N-1} \mathcal{A}_m \cos(-\omega(k_m)t + \varphi_m)\, \sin(k_m j a) \tag{6}$$

with

$$k_m a = \pi\left(\frac{\frac{1}{2}+m}{N+\frac{1}{2}}\right) = \pi\left(\frac{1+2m}{2N+1}\right) \tag{7}$$

and

$$m = 0 \ldots N-1 \tag{8}$$

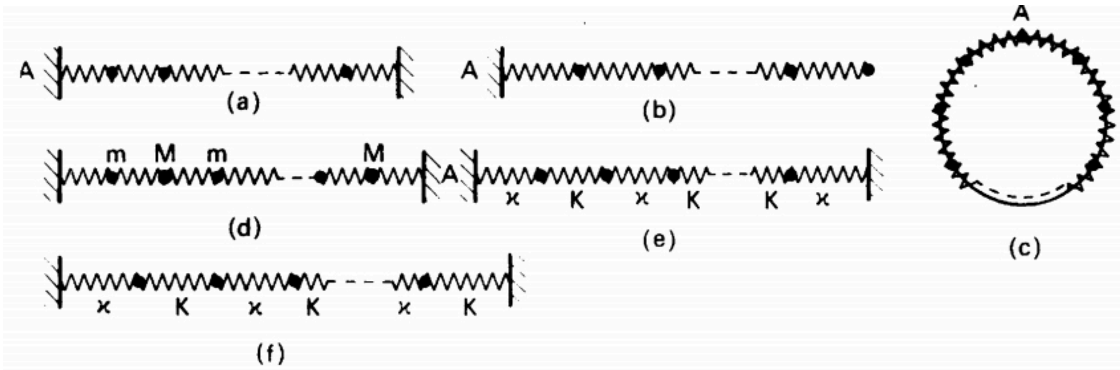

FIG. 33

Figure 2: A number of figures for the problems on springs

Problem 4. Group velocity of a chain from a continuum theory

(a) Determine the frequencies of the eigen-vibrations of a system of $2N$ particles, alternating with masses m and M, connected by springs of elastic constant γ and separation a. This is similar to the problem shown in Fig. 33d above, but we will we assume periodic boundary conditions, $q_N(t) = q_{-N}(t)$

Hint: Try an ansatz

$$q_j = \xi_1 e^{i(kx_j - \omega t)} \tag{9}$$

$$q_{j+1} = \xi_2 e^{i(kx_{j+1} - \omega t)} \tag{10}$$

and find a two-by-two eigen value equation for (ξ_1, ξ_2). This gives two eigen frequencies $\omega_\pm^2(k)$ for each value of k.

You should find that eigen solutions of this equation are

$$\omega_\pm^2(k) = \frac{\gamma}{\hat{\mu}} \pm \sqrt{\left(\frac{\gamma}{\hat{\mu}}\right)^2 - \frac{4\gamma^2}{mM}\sin^2(ka)} \tag{11}$$

where the reduced mass is

$$\hat{\mu} = \frac{mM}{(m+M)} \qquad \frac{1}{\hat{\mu}} = \frac{1}{m} + \frac{1}{M} \tag{12}$$

(b) Determine the dispersion curve $\omega_\pm(k)$ at small k, $ka \ll 1$, to order k^3 (inclusive) and sketch $\pm\omega_+(k)$ and $\pm\omega_-(k)$ at small k on the same graph. Determine the group velocity to order k^2.