Problem 1. Equations of motion

(a) From the Euler-Lagrange equations, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - V(q)$$

where $V(q) = \lambda q^4$. Also determine the canonical stress tensor for this action.

(b) By varying the action, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha q \partial_x^4 q$$

(c) By varying the action, determine the partial differential equation of motion resulting from the following action:

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha(\partial_x^2 q)^2$$

Compare to part (b) and comment on similarities and differences.

Solution:

(a) The equation of motion from the Euler lagrange equation is

$$-\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \right) + \frac{\partial \mathcal{L}}{\partial q} = 0 \tag{1}$$

where we use the index notation from class class $\partial_\nu q = (\partial_t q, \partial_x q)$. The Euler-Lagrange equations read

$$-\partial_t(\mu \partial_t q) + \partial_x(T \partial_x q) - V'(q) = 0 \tag{2}$$

To write down the canonical stress tensor we use

$$T^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu q)} \partial_\nu q + \mathcal{L} \delta^\mu{}_\nu \tag{3}$$

The energy density

$$\epsilon = -T^0{}_0 = \frac{1}{2}\mu(\partial_t q)^2 + \frac{T}{2}(\partial_x q)^2 + V(q) \tag{4}$$

The energy flux is

$$S^x{}_0 = -T^x{}_0 = -T(\partial_x q)(\partial_t q) \tag{5}$$

The momentum is

$$g_x = T^0{}_x = -\mu(\partial_t q)(\partial_x q) \tag{6}$$

The momentum flux is

$$T^x{}_x = -\frac{T}{2}(\partial_x q)^2 - \frac{\mu}{2}(\partial_t q)^2 + V(q) \tag{7}$$

(b) To find the equation of motion in these cases we vary the action sending

$$q \to q + \delta q \tag{8}$$

Then substituting $q + \delta q$ into the action and expanding to first order we have:

$$S[q + \delta q] = S[q] + \int dt dx \left[\mu(\partial_t q)(\partial_t \delta q) - T(\partial_x q)(\partial_x \delta q) - \alpha \delta q \partial_x^4 q - \alpha q \partial_x^4 \delta q \right] . \tag{9}$$

Now we integrate by parts as many times as necessary so that δq is by itself

$$T\partial_x q \, \partial_x \delta q \to [-\partial_x(T\partial_x q)] \, \delta q \, , \tag{10}$$

$$\alpha q \, \partial_x^4 \delta q \to \left[\alpha \partial_x^4 q \right] \delta q \, , \tag{11}$$

yielding

$$S[q + \delta q] = S[q] + \int dt dx \; \delta q(t, x) \; \left[-\partial_t(\mu \partial_t q) + \partial_x(T\partial_x q) - 2\alpha \partial_x^4 q \right] . \tag{12}$$

Thus the equation of motion is

$$\left[-\partial_t(\mu \partial_t q) + \partial_x(T\partial_x q) - 2\alpha \partial_x^4 q \right] = 0 \tag{13}$$

(c) We notice that the after integrating by parts twice in the action

$$\alpha(\partial_x^2 q)(\partial_x^2 q) \to \alpha q \partial_x^4 q\,, \tag{14}$$

The action is then identical with the previous item (b). It therefore gives the same EOM.

Problem 2. Group velocity

(a) (Optional – Warm up) This is lecture material, you can just copy the derivation, or skip it. Find the eigen-frequencies and normal modes of N particles which are connected by springs and which can move along a circle, *i.e* periodic boundary conditions $q_{N/2}(t) = q_{-N/2}(t)$. All particles have mass m and are separated by a distance a; the spring constants of all of the springs are the same and equal γ.

Express a general real solution $q_\ell(t)$ with $\ell = 1 \dots N$ as a linear superposition of the eigen-modes.

(b) Explain why the work done per time by the $j-1$th mass on the j-th mass is

$$\frac{dW}{dt} = -\gamma \dot{q}_j q_j + \gamma \dot{q}_j q_{j-1}$$

(c) Show that if the motion is that of a wave traveling along the circle $q_j = \mathcal{A}e^{ikx-i\omega(k)t}$, the time averaged energy flux (i.e. the time average of the work in (b)) equals the product of the (time averaged) energy per site and the group velocity of the system.

Here are some intermediate steps:

(i) (Optional, but recommended.) For computing the averages, prove the following result: if $A(t) = \mathrm{Re}[A_\omega e^{-i\omega t}]$ and $B(t) = \mathrm{Re}[B_\omega e^{-i\omega t}]$ then

$$\overline{A(t)B(t)} = \frac{1}{4}\left(A_\omega B_\omega^* + A_\omega^* B_\omega\right) = \frac{1}{2}\mathrm{Re}[A_\omega B_\omega^*] \tag{15}$$

Prove this result for yourself, by writing $A(t)$ as $\frac{1}{2}(A_\omega e^{-i\omega t} + A_\omega^* e^{+i\omega t})$.

We will use this below where $A_\omega = \mathcal{A}e^{ikx}$ or something similar.

(ii) Show that the time averaged kinetic energy per site and the time averaged potential energy per site are

$$\overline{T} = \frac{1}{4}m\omega^2(k)|\mathcal{A}|^2 = \frac{1}{4}\gamma|\mathcal{A}|^2(2\sin(ka/2))^2 \tag{16}$$

$$\overline{U} = \frac{1}{4}\gamma|\mathcal{A}|^2(2\sin(ka/2))^2 \tag{17}$$

(iii) Complete task (c) described above.

Solution:

(a) The equation of motion is the same as derived in class

$$m\ddot{q}_j + \underbrace{\kappa(q_{j+1} - q_j)}_{\text{force } F_+} + \underbrace{-\kappa(q_j - q_{j-1})}_{\text{force } F_-} = 0\,. \tag{18}$$

The two terms have a simple interpretation. The first term is the force by the $j+1$ particle on the j-th particle. The second term is the force by the $j-1$ particle on the j-th. The equilibrium position of the particles (as measured along the circumference) is $x_j = ja$ with $j = 0 \ldots N-1$. The spacing a between the particles is $a = 2\pi R/N$.

Then we substitute into the EOM $q_j(t) = e^{ikja - i\omega t}$ to find the dispersion curve $\omega(k)$ as was done in class

$$\omega(k) = \omega_0 \sin(ka/2)\,, \tag{19}$$

where $\omega_0 = 2\sqrt{\kappa/m}$. Since the particles are on a circle the N-th particle is identified with the 0-th yielding a boundary condition which ultimately quantizes k

$$q_N(t) = q_0(t)\,, \tag{20}$$

This boundary condition leads to the requirement that

$$e^{ikNa} = e^{i0} \tag{21}$$

Thus $k\,N\,a$ should be a multiple of 2π

$$k_m = \frac{m}{N}\frac{2\pi}{a} \qquad m = \text{integer} \tag{22}$$

However, as discussed in class, not all m lead to a distinct eigenvectors since

$$q_j \propto e^{ik_m ja} = e^{imj(2\pi)/N}\,. \tag{23}$$

Thus when $m = N$ this is the same as $m = 0$, since

$$e^{ij(2\pi)} = 1 \tag{24}$$

When $m = N + 1$ this is the same as $m = 1$ since

$$e^{i(N+1)j(2\pi)/N} = e^{ij(2\pi)/N} \tag{25}$$

Thus the distinct values of m are

$$k_m = \frac{m}{N}\frac{2\pi}{a} \qquad m = 0, 1, \ldots N-1 \tag{26}$$

(b) The force by the $j-1$ mass on the j-th is

$$F_- = -\kappa(q_j - q_{j-1}) \tag{27}$$

Then the work done per time is

$$\frac{dW}{dt} = -\dot{q}_j \kappa (q_j - q_{j-1}) \tag{28}$$

(c) The group velocity is

$$v_g = \frac{d\omega}{dk} = \frac{\omega_0 a}{2} \cos(ka/2) \tag{29}$$

The energy per mode is

$$T + U = \sum_j \frac{1}{2} m \dot{q}_j^2 + \frac{1}{2} \kappa (q_{j+1} - q_j)^2 \tag{30}$$

leading us to evaluate $\dot{q}_j$ and $q_{j+1} - q_j$. If $q_j(t, x) = \mathcal{A} e^{i(kja - \omega t)}$ then

$$\dot{q}_j = -i\omega \mathcal{A} e^{i(kja-\omega t)} \tag{31}$$

where ω is given by Eq. (19). The difference in displacements is

$$q_{j+1} - q_j = \mathcal{A} e^{i(k(j+1)a - \omega t)} - \mathcal{A} e^{i(kja-\omega t)} \tag{32}$$

$$= \mathcal{A} e^{i(kja-\omega t)} (e^{ika} - 1) \,. \tag{33}$$

Now we need to recall the averaging procedure over time for harmonic quantities. If $A(t) = \mathrm{Re}[A_\omega e^{-i\omega t}]$ and $B(t) = \mathrm{Re}[B_\omega e^{-i\omega t}]$, then

$$\overline{A(t)B(t)} = \frac{1}{2} \mathrm{Re}[AB^*] \tag{34}$$

In this way

$$\overline{\frac{1}{2} m \dot{q}_j^2} = \frac{1}{4} m \omega^2(k) \mathcal{A}^2 \,. \tag{35}$$

And, using that $|e^{ika} - 1|^2 = (2\sin^2(ka/2))^2$ we see that

$$\overline{\frac{1}{2} \kappa (q_{j+1} - q_j)^2} = \frac{1}{4} \kappa \mathcal{A}^2 (2\sin(ka/2))^2 \,. \tag{36}$$

Using the dispersion relation in Eq. (19)

$$\omega^2(k) = \frac{\kappa}{m} (2\sin(ka/2))^2 \,, \tag{37}$$

the total average energy density is

$$\overline{T + U} = \frac{1}{2} \kappa \mathcal{A}^2 (2\sin(ka/2))^2 \,. \tag{38}$$

The energy density times the group velocity is

$$(\overline{T + U}) v_g = \kappa \mathcal{A}^2 \, \omega_0 a (\sin(ka/2))^2 \cos(ka/2) \,. \tag{39}$$

For comparison below, we measure group velocity in terms of lattice sites per time $\hat{v}_g = v_g/a$. Thus the average energy transmitted across a lattice site per time is

$$(\overline{T+U})\hat{v}_g = \kappa\mathcal{A}^2\,\omega_0(\sin(ka/2))^2\cos(ka/2)\,. \tag{40}$$

Now we want to compare to the average work done by the $j-1$ site on the j site

$$\overline{\frac{dW}{dt}} = \overline{-\kappa\dot{q}_j(q_j - q_{j-1})} \tag{41}$$

Then as before

$$\dot{q}_j = \mathcal{A}\,e^{i(kja-\omega t)}(-i\omega)\,, \tag{42}$$

and

$$q_j - q_{j-1} = \mathcal{A}\,e^{i(kja-\omega t)}(1-e^{-ika})\,. \tag{43}$$

So using the averaging theorem again we find:

$$\begin{aligned}\overline{\frac{dW}{dt}} =& -\frac{1}{2}\kappa\mathcal{A}^2\,\mathrm{Re}\left[(-i\omega)(1-e^{-ika})^*\right] && (44)\\ =& -\frac{1}{2}\kappa\mathcal{A}^2\,\mathrm{Re}\left[-i\omega\left(2ie^{-ika/2}\sin(ka/2)\right)^*\right] && (45)\\ =&\kappa\mathcal{A}^2\,\omega(k)\cos(ka/2)\sin(ka/2) && (46)\\ =&\kappa\mathcal{A}^2\omega_0\cos(ka/2)\sin^2(ka/2) && (47)\end{aligned}$$

This agrees with Eq. (40)

Solution:

(a)

The equilibrium positions of the particles are

$$x_j = ja \qquad j = 1 \ldots N \tag{48}$$

and q_j denotes the displacements of the j-th lattice site. We will consider two fictitious particles at $x_0 = 0$ and x_{N+1} which are not allowed to move

Then we substitute

$$q_j = \mathcal{A}e^{ikja - \omega(k)t} \tag{49}$$

and determine the dispersion curve as usual. The result is the Debye dispersion curve $\omega(k)$ given in Eq. (19).

Now however the boundary conditions at x_0 namely

$$q_0 = 0 \tag{50}$$

can only be satisfied at all times if we takes a superposition of these waves

$$q_j(t) = \mathcal{A}\left[e^{ikja - \omega(k)t} - Ae^{-ikja - \omega(k)t}\right] \tag{51}$$

$$q_j(t) = \mathcal{A}e^{-i\omega t}\sin(kja) \tag{52}$$

Here it is understood that we are to take the real part of these expressions.

For any arbitrary k the boundary condition at x_{N+1}

$$q_{N+1} = 0 \tag{53}$$

will not be satisfied. Only when $k(N+1)a = m\pi$ (with m integer) will the boundary conditions be satisfied

$$k_m = \frac{\pi}{a}\frac{m}{N+1} \tag{54}$$

In fact not all values of m lead to distinct normal modes. The allowed values are $m = 1 \ldots N$. This provides an independent set of normal modes

$$\boxed{k_m = \frac{\pi}{a}\frac{m}{N+1} \qquad m = 1 \ldots N} \tag{55}$$

To summarize our general solution for the m-th normal mode is

$$q_j(t) = \mathrm{Re}[\mathcal{A}e^{-i\omega(k_m)t}]\sin(k_m ja) = \mathcal{A}\cos(-\omega(k_m)t + \varphi)\sin(k_m ja) \tag{56}$$

The general solution is a superposition of these normal modes

$$\boxed{q_j(t) = \sum_{m=1}^{N}\mathcal{A}_m\cos(-\omega(k_m)t + \varphi_m)\sin(k_m ja)} \tag{57}$$

(b) When one end is fixed and one end is free the solution is analogous to (a); Eq. (68) remains valid. Now, however, the condition of free end point motion is that the force on the N-th particle by a fictitious $N+1$ particle should be zero. Thus we require

$$q_N = q_{N+1} \tag{58}$$

In the continuum theory this means that the derivative vanishes at the endpoints.

We need that

$$\sin(kNa) \qquad \sin(k(N+1)a) \tag{59}$$

should be equal. This can happen if the two points exactly straddle the maximum of the sin curve. Thus we want

$$k_m(N+\tfrac{1}{2})a = \frac{\pi}{2} + m\pi \tag{60}$$

This leads to

$$\boxed{k_m a = \pi\left(\frac{\frac{1}{2}+m}{N+\frac{1}{2}}\right) = \pi\left(\frac{1+2m}{2N+1}\right)} \tag{61}$$

As is usual not all k_m are distinct. The N independent normal modes can be taken to be

$$m = 0 \ldots N-1 \tag{62}$$

Thus the complete solution in this case is

$$\boxed{q_j(t) = \sum_{m=0}^{N-1} \mathcal{A}_m \cos(-\omega(k_m)t + \varphi_m)\,\sin(k_m ja)} \tag{63}$$

newpage **Solution:**

(a)

The equilibrium positions of the particles are

$$x_j = ja \qquad j = 1 \ldots N \tag{64}$$

and q_j denotes the displacements of the j-th lattice site. We will consider two fictitious particles at $x_0 = 0$ and x_{N+1} which are not allowed to move

Then we substitute

$$q_j = \mathcal{A}e^{ikja-\omega(k)t} \tag{65}$$

and determine the dispersion curve as usual. The result is the Debye dispersion curve $\omega(k)$ given in Eq. (19).

Now however the boundary conditions at x_0 namely

$$q_0 = 0 \tag{66}$$

can only be satisfied at all times if we takes a superposition of these waves

$$q_j(t) = \mathcal{A}\left[e^{ikja-\omega(k)t} - Ae^{-ikja-\omega(k)t}\right] \tag{67}$$

$$q_j(t) = \mathcal{A}e^{-i\omega t}\sin(kja) \tag{68}$$

Here it is understood that we are to take the real part of these expressions.

For any arbitrary k the boundary condition at x_{N+1}

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will not be satisfied. Only when $k(N+1)a = m\pi$ (with m integer) will the boundary conditions be satisfied

$$k_m = \frac{\pi}{a}\frac{m}{N+1} \tag{70}$$

In fact not all values of m lead to distinct normal modes. The allowed values are $m = 1 \ldots N$. This provides an independent set of normal modes

$$\boxed{k_m = \frac{\pi}{a}\frac{m}{N+1} \qquad\qquad m = 1\ldots N} \tag{71}$$

To summarize our general solution for the m-th normal mode is

$$q_j(t) = \mathrm{Re}[\mathcal{A}e^{-i\omega(k_m)t}]\,\sin(k_m ja) = \mathcal{A}\cos(-\omega(k_m)t + \varphi)\sin(k_m ja) \tag{72}$$

The general solution is a superposition of these normal modes

$$\boxed{q_j(t) = \sum_{m=1}^{N}\mathcal{A}_m\cos(-\omega(k_m)t + \varphi_m)\sin(k_m ja)} \tag{73}$$

(b) When one end is fixed and one end is free the solution is analogous to (a); Eq. (68) remains valid. Now, however, the condition of free end point motion is that the force on the N-th particle by a fictitious $N+1$ particle should be zero. Thus we require

$$q_N = q_{N+1} \tag{74}$$

In the continuum theory this means that the derivative vanishes at the endpoints.

We need that

$$\sin(kNa) \qquad \sin(k(N+1)a) \tag{75}$$

should be equal. This can happen if the two points exactly straddle the maximum of the sin curve. Thus we want

$$k_m(N+\tfrac{1}{2})a = \frac{\pi}{2} + m\pi \tag{76}$$

This leads to

$$\boxed{k_m a = \pi\left(\frac{\frac{1}{2}+m}{N+\frac{1}{2}}\right) = \pi\left(\frac{1+2m}{2N+1}\right)} \tag{77}$$

As is usual not all k_m are distinct. The N independent normal modes can be taken to be

$$m = 0\ldots N-1 \tag{78}$$

Thus the complete solution in this case is

$$\boxed{q_j(t) = \sum_{m=0}^{N-1}\mathcal{A}_m\cos(-\omega(k_m)t + \varphi_m)\,\sin(k_m ja)} \tag{79}$$

Problem 3. Group velocity of a chain from a continuum theory

(a) Determine the frequencies of the eigen-vibrations of a system of $2N$ particles, alternating with masses m and M, connected by springs of elastic constant γ and separation a. This is similar to the problem shown in Fig. 33d above, but we will we assume periodic boundary conditions, $q_N(t) = q_{-N}(t)$

Hint: Try an ansatz

$$q_j = \xi_1 e^{i(kx_j - \omega t)} \tag{80}$$

$$q_{j+1} = \xi_2 e^{i(kx_{j+1} - \omega t)} \tag{81}$$

and find a two-by-two eigen value equation for (ξ_1, ξ_2). This gives two eigen frequencies $\omega_\pm(k)$ for each value of k.

(b) Determine the dispersion curve $\omega_\pm(k)$ at small k, $ka \ll 1$, to order k^3 and sketch $\omega_+(k)$ and $\omega_-(k)$ at small k on the same graph. Determine the group velocity to order k^2.

(c) When the wavelength of the waves of part (a) is very long, the microscopic details of the discrete model in (a), are unimportant. A continuum theory can reproduce the results of the model in (a), provided the "low energy" constants of the continuum theory are adjusted to match certain physical properties.

Consider the action from a previous problem

$$S = \int dt\, dx \frac{1}{2}\mu(\partial_t q)^2 - \frac{T}{2}(\partial_x q)^2 - \alpha(\partial_x^2 q)^2 \tag{82}$$

From the equation of motion you found previously, determine the dispersion curve $\omega(k)$ associated with this action. What should the values of the "low-energy constants", μ, T and α be set to if the continuum action in Eq. (82), is to reproduce the dispersion curve of the discrete theory of parts (a) and (b) at small k for the "plus" modes (i.e. the modes with eigenfrequencies $\omega_+(k)$ in the discrete theory) and how should they be tuned to reproduce the "minus" modes.

Solution:

(a) We have to work out the Lagrangian

$$L = \sum_{j=1}^{N} \frac{1}{2} m\dot{q}_{2j-1}^2 + \frac{1}{2} M\dot{Q}_{2j}^2 - \frac{1}{2}\kappa(Q_{2j} - q_{2j-1})^2 - \frac{1}{2}\kappa(q_{2j-1} - Q_{2j-2})^2 - \frac{1}{2}\kappa(q_{2j+1} - Q_{2j})^2 \quad (83)$$

The equations of motion which follows here for the small mass objects are

$$m\ddot{q}_{2j-1} = \kappa(Q_{2j} - q_{2j-1}) - \kappa(q_{2j-1} - Q_{2j-2}) \quad (84)$$

$$m\ddot{q}_{2j-1} = \kappa(Q_{2j} - 2q_{2j-1} + Q_{2j-2}) \quad (85)$$

while the heavy objects satisfy

$$M\ddot{Q}_{2j} = -\kappa(Q_{2j} - q_{2j-1}) + \kappa(q_{2j+1} - Q_{2j}) \quad (86)$$

$$M\ddot{Q}_{2j} = \kappa(q_{2j+1} - 2Q_{2j} + q_{2j-1}) \quad (87)$$

Now we try a wave form

$$q_{2j-1} = \mathcal{A}_1 e^{i(k(2j-1)a + \omega t)} \quad (88)$$

$$Q_{2j} = \mathcal{A}_2 e^{i(k(2j)a + \omega t)} \quad (89)$$

Plugging it in to the equations of motion we find an eigenvalue equation the allowed frequencies:

$$-\omega^2 \begin{pmatrix} m & 0 \\ 0 & M \end{pmatrix} \begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{pmatrix} = \begin{pmatrix} -2\kappa & \kappa(e^{ika} + e^{-ika}) \\ \kappa(e^{ika} + e^{-ika}) & -2\kappa \end{pmatrix} \begin{pmatrix} \mathcal{A}_1 \\ \mathcal{A}_2 \end{pmatrix} \quad (90)$$

The eigen solutions of this equation are

$$\omega^2(k) = \frac{\kappa}{\mu} \pm \sqrt{\left(\frac{\kappa}{\mu}\right)^2 - \frac{4\kappa^2}{mM}\sin^2(ka)} \quad (91)$$

where the reduced mass is

$$\hat{\mu} = \frac{mM}{(m+M)} \qquad \frac{1}{\hat{\mu}} = \frac{1}{m} + \frac{1}{M} \quad (92)$$

We will leave it here as it is sufficient for the next problem.

We expand the eigen frequencies at small k, keeping only ω_- whose frequency approaches zero as $k \to 0$. This is the only wavelike solutions at small k. ω_+ approaches a constant as $k \to 0$. ω_- has the expansion at small k of the form

$$\omega_-^2 = \frac{2a}{M+m}(a\kappa)k^2 + \frac{2a}{M+m}(a\kappa)\left[\frac{-1}{3} + \frac{mM}{(M+m)^2}\right](a^2k^4) \quad (93)$$

Let us introduce the mass per length μ, the Youngs modulus κa, and the phase velocity v_0, as was done for the simple chain in lecture

$$\mu \equiv \frac{M+m}{2a} \tag{94}$$

$$Y \equiv \kappa a \tag{95}$$

$$v_0^2 \equiv \frac{Y}{\mu} \tag{96}$$

With this notation the dispersion curve reads

$$\boxed{\omega_-^2 = v_0^2 k^2 - v_0^2 \left(\frac{1}{3} - \frac{mM}{(M+m)^2}\right) a^2 k^4 + \ldots} \tag{97}$$

Taking a square root we have

$$\omega_-(k) = v_0 k \left(1 - \frac{a^2k^2}{2}\left(\tfrac{1}{3} - \frac{mM}{(M+m)^2}\right) + \ldots\right) \tag{98}$$

and the group velocity is

$$v_g = v_0 \left(1 - \frac{3a^2k^2}{2}\left(\tfrac{1}{3} - \frac{mM}{(M+m)^2}\right) + \ldots\right) \tag{99}$$

(b) The wave equation from this action is

$$\left[-\partial_t(\mu\partial_t q) + \partial_x(T\partial_x q) - 2\alpha\partial_x^4 q\right] = 0 \tag{100}$$

The mass per length and the Tension/Youngs Modulus are identified with

$$\mu = \frac{M+m}{2a}\,. \tag{101}$$

$$T = Y = \kappa a\,. \tag{102}$$

Substituting $e^{-i\omega t + ikx}$ into the wave equation we find that the dispersion curve is

$$\omega^2 - v_0^2 k^2 - \frac{2\alpha}{\mu} k^4 = 0 \tag{103}$$

So in order for the continuum theory to reproduce the microscopic physics we require

$$\frac{2\alpha}{\mu} k^4 = -v_0^2 \left(\frac{1}{3} - \frac{mM}{m+M}\right) a^2 k^4 \tag{104}$$

Thus the low energy constant α is for this microscopic theory

$$\alpha = -\frac{(Ya^2)}{2}\left(\frac{1}{3} - \frac{mM}{m+M}\right) \tag{105}$$