Problem 1. Small Extensions of Last Homework:

(a) See Homework 14, Solutions, Problem 4. Part (c) has been added. Closely related to Homework 14 Problem 4 are problems that appear on the 2019 and 2020 finals.

  (i) See Final 2019 problem 4.

  (ii) See Final 2020 problem 4.

(b) See Homework 14, Solutions, Problem 1. Part (a) has been extended by asking you to determine the canonical stress tensor.

(c) See Homework 14, Solutions, Problem 3. Part (b) was optional but it is recommended.

Problem 2. Split personality

This problem discusses wave packets.

(a) A general solution to the wave equation is

$$y(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[A(k) e^{i(kx-\omega(k)t)} + B(k) e^{i(kx+\omega(k)t)} \right] \tag{1}$$

where $\omega(k)$ is a positive symmetric function of k, $\omega(-k) = \omega(k)$. For a real wave $B(-k)$ must be equal to $A^*(k)$. By change of variables $k \to -k$ in the second integral the solution can be written[1]

$$y(t,x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[A(k) e^{i(kx-\omega(k)t)} + A^*(k) e^{-i(kx-\omega(k)t)} \right] \tag{2}$$

The wave equation is a second order differential equation. Thus in order to specify the problem, we need to specify the initial amplitude $y(0,x)$ and the initial velocity $\partial_t y(0,x)$ everywhere on the string. How is $A(k)$ determined by $y(0,x)$ and $\partial_t y(0,x)$?

(b) Here we want to describe a wave-packet which moves to the right. The amplitude at $t=0$ is

$$y(0,x) = \mathrm{Re}[g(x)e^{ik_0 x}]\,, \tag{3}$$

with

$$g(x) = \frac{1}{\sqrt{2\pi a^2}} \exp(-x^2/(2a^2))\,, \tag{4}$$

and $k_0 a \gg 1$. Argue that the appropriate initial condition for a right moving wave is

$$\partial_t y \simeq v_\phi \, \partial_x y \tag{5}$$

[1] Having had this discussion with the grad-students in the past ... $\int_{-\infty}^{\infty} dk f(k) = \int_{+\infty}^{-\infty} -d\tilde{k} f(-\tilde{k}) = \int_{-\infty}^{\infty} d\tilde{k} f(-\tilde{k})$, and then since $\tilde{k}$ is a dummy integration variable, we now just call it k to arrive at the result Eq. (2)

where $v_\phi(k_0) = \omega(k_0)/k_0$ is the phase velocity of the wave, by (approximately) computing $A(k)$ in this case. What would $A(k)$ be if $\partial_t y(0, x) = 0$? Sketch $|A(k)|^2$ in both cases.

In the second case $\partial_t y = 0$, one can either calculate the result directly or use the superposition principle.

(c) Repeat the argument (given in class for complex waves) that if the solution for a wave is

$$y(t, x) = \int \frac{dk}{2\pi} \left[A(k) e^{i(kx - \omega(k)t)} + A^*(k) e^{-i(kx - \omega(k)t)} \right] , \tag{6}$$

then, provided the wave form is initialized as in (5), then

$$y(t, x) \simeq \cos(k_0 x - \omega_0 t)\, g(x - Ut) \,. \tag{7}$$

Here $U = d\omega(k_0)/dk$ is the group velocity and $\omega_0 = \omega(k_0)$. The applet by Michael Fowler is a helpful visualization.

(d) Determine the wave form at late times if $\partial_t y(0, x) = 0$. Hint: use the superposition principle.

Solution:

(a) We have to epress the real and imaginary parts of $A(k)$ with $y(0,x)$ and $\dot{y}(0,x)$. We have

$$y(0,x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}(A(k)e^{ikx} + A^*(k)e^{-ikx}) \tag{8}$$

The second integral is transformed to

$$y(0,x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}(A(k) + A^*(-k))e^{ikx} \tag{9}$$

Then similarly

$$\dot{y}(t,0) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\,(-i\omega(k))(A(k) - A^*(-k))e^{ikx} \tag{10}$$

From these expressions

$$A(k) + A^*(-k) = \int_x e^{-ikx} y(0,x) \tag{11}$$

$$-i\omega(k)(A(k) - A^*(-k)) = \int_x e^{-ikx}\dot{y}(0,x) \tag{12}$$

Putting together the ingrediants

$$A(k) = \int_x e^{-ikx}\left(y(0,x) + \frac{i}{\omega(k)}\dot{y}(0,x)\right) \tag{13}$$

(b) For reference we note the following transform pairs

$$g(x) \equiv \frac{1}{\sqrt{2\pi a^2}}\exp(-x^2/2a^2) \leftrightarrow G(k) \equiv \exp(-k^2a^2/2) \tag{14}$$

We wish to use part (a). The real part of is

$$y(0,x) = g(x)\cos(k_0x) = \frac{1}{2}g(x)e^{ik_0x} + \frac{1}{2}g(x)e^{-ik_0x} \tag{15}$$

and we have

$$\partial_x y(0,x) \simeq -k_0 g(x)\sin(k_0x)\,. \tag{16}$$

If

$$\dot{y}(0,x) \simeq v_\phi(k_0)\partial_x y\,, \tag{17}$$

then

$$\dot{y}(0,x) \simeq -\omega(k)g(x)\sin(k_0x)\,. \tag{18}$$

In this way we find

$$\left(y(0,x) \pm \frac{i}{\omega(k_0)}\dot{y}(0,x)\right) \simeq g(x)\left(\cos(k_0x) \mp i\sin(k_0x)\right) = g(x)e^{-ik_0x} \tag{19}$$

And so the function $A(k)$ is peaked near the positive value $k \simeq k_0$

$$\boxed{A(k) = \frac{1}{2}\int_x e^{i(k-k_0)x} g(x) \simeq \frac{1}{2}\exp(-(k-k_0)^2 a^2/2)} \tag{20}$$

indicating that it is primarily a right mover.

If we had $\dot{y}(0,x) = 0$ then we would find that

$$\boxed{A(k) = \frac{1}{4}\exp(-(k-k_0)^2 a^2/2) + \frac{1}{4}\exp(-(k+k_0)^2 a^2/2)} \tag{21}$$

Then the function $A(k)$ would have peaks at $k = k_0$ and $k = -k_0$. It descrbes a 50/50 superposition of right and left movers.

(c) Then the full solution takes the form

$$y(t,x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dk}{2\pi}\, A(k) e^{ik(x-\omega(k)t)} + \text{c.c.} \tag{22}$$

where c.c. denotes the complex conjugate of the first term:

$$y(t,x) = \underbrace{\frac{1}{2}\int_{k\sim k_0} \frac{dk}{2\pi}\, A(0,k) e^{ik(x-\omega(k)t)}}_{I_+} + I_+^* \tag{23}$$

$A(0,k)$ is sharply peaked near $k = k_0$. Near $k = k_0$ the frequency is expanded

$$\omega(k) \simeq \omega_0 + \left.\frac{d\omega}{dk}\right|_{k_0} \tilde{k} \tag{24}$$

where $\omega_0 \equiv \omega(k_0)$ and $\tilde{k} = (k - k_0)$. The overall phase is

$$(kx - \omega(k)t) \simeq (k_0 x - \omega_0 t) + \tilde{k}(x - v_g t) \tag{25}$$

where the group velocity

$$v_g = \left.\frac{d\omega}{dk}\right|_{k_0} \tag{26}$$

The wave from the integration near k_0 is then

$$I_+ = e^{ik_0 x - \omega_0 t}\int \frac{d\tilde{k}}{2\pi} G(\tilde{k}) e^{i\tilde{k}(x-v_g t)} \tag{27}$$

The first integral evaluates to

$$I_+ = \frac{1}{2} e^{i(k_0 x - \omega_0 t)} g(x - v_g t) \tag{28}$$

The combination of $I_+ + I_+^*$ appearing in Eq. (23) is

$$y(t,x) = \cos(k_0 x - \omega_0 t)\, g(x - v_g t) \tag{29}$$

showing that the overall wave packet (envelope) moves with the group velocity, while the phase moves with the phase velocity $v_0 = \omega_0/k_0$:

$$\boxed{y(t,x) = \cos(k_0(x - v_0 t))\, g(x - v_g t)} \tag{30}$$

At $t = 0$ we recover the original wave form

$$y(0,x) = \cos(k_0 x)\, g(x)\,. \tag{31}$$

If we had initially $\dot{y}(0,x)$ then we would find a superposition of two wave forms

$$\boxed{y(t,x) = \tfrac{1}{2}\cos(k_0(x - v_0 t))\, g(x - v_g t) + \tfrac{1}{2}\cos(k_0(x + v_0 t))\, g(x + v_g t)} \tag{32}$$