## Problem 1. (MIT/OCW) Spring system on a plane

A massless spring has an unstretched length $b$ and spring constant $k$, and is used to connect two particles of mass $m_1$ and $m_2$. The system rests on a frictionless table and may oscillate, translate, and rotate.

(a) What is the Lagrangian? Write it with two-dimensional cartesian coordinates $\boldsymbol{r}_1 = (x_1, y_1)$ and $\boldsymbol{r}_2 = (x_2, y_2)$. There are four coordinates in total.

(b) Setup a suitable set of generalized coordinates (four in total) to better account for the symmetries of this system. Take one of your coordinates to be $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$. What is the Lagrangian in these variables?

(c) Identify three conserved generalized momenta that are associated to cyclic coordinates in the Lagrangian from part (b). If you think you are missing some, try to improve your answer to (b). Briefly explain the physical meaning of each of the three conserved generalized momenta. Show that the equation of motion for $r$ takes the form

$$m_{\text{eff}}\ddot{r} = -\frac{\partial V_{\text{eff}}(r)}{\partial r} \tag{1}$$

with an appropriate $m_{\text{eff}}$ and $V_{\text{eff}}(r)$.

(d) Write down the hamiltonian function $h(q, \dot{q}, t)$ for the coordinates chosen in (b). Show that that the velocity $\dot{r}$ associated with the coordinate $r$ (here $r = |\boldsymbol{r}_1 - \boldsymbol{r}_2|$ is distance between the particles) can be determined from the energy $E$ and an effective potential $V_{\text{eff}}(r)$ which depends on the rotation rate, i.e. show that

$$\frac{1}{2}m_{\text{eff}}\dot{r}^2 + V_{\text{eff}}(r) = E \tag{2}$$

(e) By examining the effective potential and its dependence on the rotation rate, show that there is a solution that rotates but does not oscillate, and discuss what happens to this solution for an increased rate of rotation. (A closed form solution is not necessary. A graphical explanation based on the effective potential will suffice.)

**Solution:**

(a) See below

(b) The Lagrangian is $L = T - V$

$$L = \frac{1}{2}m_1\dot{\boldsymbol{r}}_1^2 + \frac{1}{2}m_2\dot{\boldsymbol{r}}_2^2 - U(|\boldsymbol{r}_1 - \boldsymbol{r}_2|) \tag{3}$$

Here $U(r) = \frac{1}{2}k(r-b)^2$. In passing to the second line we have defined the cente of mass and relative coordinates

$$M \equiv m_1 + m_2 \tag{4}$$
$$\mu \equiv m_1 m_2/M \tag{5}$$
$$\boldsymbol{R} \equiv (m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2)/M \tag{6}$$
$$\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2 \tag{7}$$

With the more explicit forms we have

$$\boldsymbol{R} = (X, Y) \qquad \boldsymbol{r} = r(\cos\phi, \sin\phi) \tag{8}$$

The Lagrangian becomes

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(r^2 + r^2\dot{\phi}^2) - U(r) \tag{9}$$

(c) There are three cylic coordinates $X, Y, \phi$ leading to three conserved quantities

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X} \tag{10}$$
$$p_Y = \frac{\partial L}{\partial \dot{Y}} = M\dot{Y} \tag{11}$$
$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu r^2\dot{\phi} \tag{12}$$

These are the linear momenta of the center of mass and the angular momentum of the system. The only non-trivial equation of motion is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r} \tag{13}$$
$$\mu\ddot{r} = \mu r\dot{\phi}^2 - \frac{\partial U}{\partial r} \tag{14}$$

Noting that $p_\phi$ is constant, the $\mu r^2\dot{\phi}$ term is

$$\mu r^2\dot{\phi}^2 = \frac{p_\phi^2}{\mu r^3} \tag{15}$$
$$= -\frac{\partial}{\partial r}\left(\frac{p_\phi^2}{2\mu r^2}\right), \tag{16}$$

leading to the overall EOM:

$$\mu\ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r} \qquad V_{\text{eff}}(r, p_\phi) \equiv \frac{p_\phi^2}{2\mu r^2} + U(r)\,. \tag{17}$$

(d) The hamiltonian function is

$$h = p_X \dot{X} + p_y \dot{Y} + p_\phi \dot{\phi} + p_r \dot{r} - L \tag{18}$$

$$= \frac{1}{2} M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}\mu(\dot{r}^2 + \dot{\phi}^2) + U(r) \tag{19}$$

When the hamiltonian function is evaluated using the equation of motion, we have $h = E$ and $p_X, p_Y$ and $p_\phi$ are constants, leading to

$$\epsilon \equiv E - \frac{p_X^2}{2m} + \frac{p_Y^2}{2m} \tag{20}$$

$$\epsilon = \frac{1}{2}\mu \dot{r}^2 + V_{\text{eff}}(r, p_\phi) \tag{21}$$

(e) A plot of the $V_{\text{eff}}$ is shown below. At the minimum of the effective potential (were $\partial V / \partial r = 0$) the system rotates without oscillations, since

$$\mu \ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r} \tag{22}$$

As the system rotates faster and faster, the non-oscillating spring stretches more and more counterbalancing the centrifugal force.

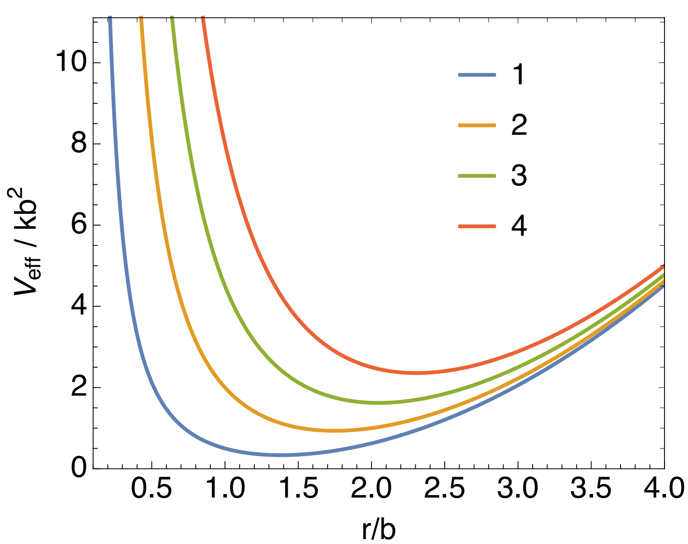


Figure 1: A plot with units $m = k = b = 1$ for different values of $p_\phi / m\omega_0 b$ with $\omega_0 = \sqrt{k/m}$.

## Problem 2. (Goldstein/MIT OCW) Jerky Mechanics

Consider an extension of classical mechanics where the equation of motion involves a triple time derivative, $\dddot{x} = f(x, \dot{x}, \ddot{x}, t)$. Lets use the action principle to derive the corresponding Euler-Lagrange equations. Start with a Lagrangian of the form $L(q^i, \dot{q}^i, \ddot{q}^i, t)$ for $n$ generalized coordinates $q^i$, and make use of the action principle for paths $q^i(t)$ that have zero variation of both $q^i$ and $\dot{q}^i$ at the end points. Show that

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}^i}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \frac{\partial L}{\partial q^i} = 0 \tag{23}$$

for each $i = 1 \ldots n$

**Solution:**

Sending $q^i \to q^i + \delta q^i$ the action $S[q]$ changes as

$$\delta S = \int dt \left( \frac{\partial L}{\partial \ddot{q}} \frac{d^2 \delta q}{dt^2} + \frac{\partial L}{\partial \dot{q}} \frac{d\delta q}{dt} + \frac{\partial L}{\partial q} \delta q \right) . \tag{24}$$

We integrate by parts twice for the $\ddot{q}$ terms yielding

$$\delta S = \left. \frac{\partial L}{\partial \ddot{q}} \delta \dot{q} - \frac{d}{dt}\left(\frac{\partial L}{\partial \ddot{q}}\right) \delta q + \frac{\partial L}{\partial \dot{q}} \delta q \right|_{t_1}^{t_2} + \int dt \left( \frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} \right) \delta q \, . \tag{25}$$

The bourndary term vanish by the constraints at the ends, e.g. $\delta q(t_1) = \delta \dot{q}(t_1) = 0$. Then since $\delta q$ is otherwise arbitrary this yields the EOM as claimed

$$\frac{d^2}{dt^2}\left(\frac{\partial L}{\partial \ddot{q}}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) + \frac{\partial L}{\partial q} = 0 \, . \tag{26}$$

## Problem 3. Equivalent Lagrangians

(a) (Goldstein) Let $L(q, \dot{q}, t)$ be the Lagrangian for a particle with coordinate $q$, which satisfies the Euler-Lagrange equations. Show that the Lagrangian

$$L' = L + \frac{dF(q,t)}{dt} \tag{27}$$

yields the same Euler-Lagrange equations as $L$ where $F$ is an arbitrary differentiable function. Give a proof based on and the action principle. We say that $L$ and $L'$ are equivalent. (If you feel like it you might also like to check directly that the EOM are the same.)

(b) (Goldstein) Using the previous problem (Problem 3), what is the equation of motion resulting from

$$L = -\frac{1}{2}mq\ddot{q} - \frac{1}{2}m\omega_0^2 q^2 \tag{28}$$

and what is it related to? Explain why this equation of motion is obvious from the Lagrangian in Eq. (28) and the result of part (a).

(c) Consider the action of a free particle

$$S[\boldsymbol{r}(t)] = \int \mathrm{d}t\, C\boldsymbol{v}^2 \tag{29}$$

where $C = m/2$ is a constant normally associated with the mass. Show that the action is unchanged (up to boundary terms) by a Galilean transformation, and hence the transformed version gives the same EOM. If the Lagrangian took the form $L = Cv^4$ this would not have been the case. Thus requiring Gallilean invariance fixes the form the velocity dependent action to involve the kinetic energy.

(d) Consider a fricitionless block of mass $m$ in one dimension. The block sits on a train, which accelerates with constant acceleration $a_0$. The block experiences no forces, and thus the action of the block is simply the free one

$$S = \int \mathrm{d}t \frac{1}{2}mv_g^2\,, \tag{30}$$

where $v_g(t)$ is the velocity relative to the ground. Let $v(t)$ denote the velocity of the block relative to the back of the train.

(i) Write down the relation between $v(t)$ and $v_g(t)$, and substitute into Eq. (30) to determine the Lagranngian for $v(t)$.

(ii) Show that this Lagrangian is equivalent to that of a particle in a potential $U(x) = ma_0x$ where $x$ is the position of the particle relative to the back of the train, and interpret the result.

**Solution:**

(a) Clearly the the action with $L'$ only differs by an endpoint contribution

$$S'[q] = \int_{t_1}^{t_2} dt\,(L + \frac{dF}{dt}) \tag{31}$$

$$= F(q(t),t)|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\,L \tag{32}$$

$$= F(q(t),t)|_{t_1}^{t_2} + S[q] \tag{33}$$

Since $F(q,t)$ is only on the boundary, it does not change as we vary $q$. Thus the variation of the action $S'[q]$ (with the endpoints fixed) is the same the variation of $S[q]$ (with endpoints fixed) yielding the same equations of motion

$$\delta S'[q,\delta q] = \delta S[q,\delta q] = 0 \tag{34}$$

(b) We can use the equations of motion for the previous problem

$$-\frac{1}{2}m\ddot{q} + 0 + \left(-\frac{1}{2}m\ddot{q} - m\omega_0^2 q\right) = 0 \tag{35}$$

i.e. the equation of motion of the SHO

$$m\ddot{q} + m\omega_0^2 q = 0 \tag{36}$$

This is obvious, as we can integrate by parts, leaving the SHO oscillator lagrangian plus neglectable total derivs:

$$L = -\frac{1}{2}mq\ddot{q} - \frac{1}{2}m\omega_0^2 q^2 \tag{37}$$

$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^2 - \frac{d}{dt}\left(\frac{1}{2}mq\dot{q}\right) \tag{38}$$

$$= L_{\rm SHO} + \frac{d}{dt}\,(\text{ignore}-\text{me}) \tag{39}$$

(c) The action for coordinate $\boldsymbol{r}$ is

$$S[\boldsymbol{r}] = \int dt C\dot{\boldsymbol{r}}^2 \tag{40}$$

In a frame moving with velocity $\boldsymbol{u}$ to right, the coordinates in the new frame $\boldsymbol{r}'$ are related to the old coordinate $\boldsymbol{r}$ via

$$\dot{\boldsymbol{r}} \to \boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{u}\,t \tag{41}$$

or $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{u}\,t$. The action for the coordinate $\boldsymbol{r}'$ is therefore

$$S[\boldsymbol{r}'] = \int dt C\dot{\boldsymbol{r}}^2 \tag{42}$$

$$= \int dt C\dot{\boldsymbol{r}}'^2 + 2C\dot{\boldsymbol{r}}' \cdot \boldsymbol{u} + Cu^2 \tag{43}$$

$$= \int dt C\dot{\boldsymbol{r}}'^2 + \frac{d}{dt}\left(2C\boldsymbol{r}' \cdot \boldsymbol{u} + Cu^2 t\right) \tag{44}$$

$$= \int dt C\dot{\boldsymbol{r}}'^2 + \text{total derivs} \tag{45}$$

Thus the equation of motion for $\boldsymbol{r}'$ will be the same as for $\boldsymbol{r}$ since they have the same action up to a boundary term.

(d) The position of the back of the train is $\frac{1}{2}a_0 t^2$ and thus the coordinates are related

$$x_g(t) = x(t) + \tfrac{1}{2}a_0 t^2 \,. \tag{46}$$

So

$$\dot{x}_g = \dot{x} + a_0 t \tag{47}$$

We have

$$S[x] = \int \frac{1}{2}m(\dot{x} + a_0 t)^2 \tag{48}$$

$$= \int \frac{1}{2}m\dot{x}^2 + m\dot{x}a_0 t + \frac{1}{2}ma_0^2 t^2 \tag{49}$$

The last term is a independent of $x$ and can be ignored (we only care about the variation of $\delta S$ and this term is constant under the variation). The second to last term can be integrated by parts

$$S[x] = \int \frac{1}{2}m\dot{x}^2 + m\dot{x}a_0 t \tag{50}$$

$$= \int \frac{1}{2}m\dot{x}^2 - ma_0 x + \frac{d}{dt}(mxa_0 t) \tag{51}$$

$$= \int \frac{1}{2}m\dot{x}^2 - ma_0 x + (\text{total derivs}) \tag{52}$$

This (without the irrelevant total-derivs) is the action for a particle experiencing a constant force $F_x = -ma_0$, i.e. a force pushing the particle to the left, towards the back of the train. This is the expected effective force because we are viewing the motion in the coordinate system of the accelerating train.