Problem 1. Motion on a train

(a) Consider a fricitionless block of mass m in one dimension. The block sits on a train, which accelerates with constant acceleration a_0. The block experiences no forces, and thus the action of the block is simply the free one

$$S = \int \mathrm{d}t \frac{1}{2} m v_g^2 \,, \tag{1}$$

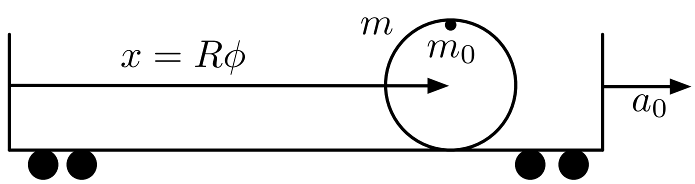
where $v_g(t)$ is the velocity of the block relative to the ground. Let $v(t)$ denote the velocity of the block relative to the back of the train.

(i) Write down the relation between $v(t)$ and $v_g(t)$, and substitute into Eq. (1) to determine the Lagranngian for $v(t)$.

(ii) Show that this Lagrangian is equivalent to that of a particle in a potential $U(x) = m a_0 x$ where x is the position of the particle relative to the back of the train, and interpret the result.

Problem 2. A cylinder on a train

Consider a cylinder-like contraption consisting of a cylindrical ring of mass m and radius R, and a small weight of mass m_0 fixed to the rim of the ring (see below). At time $t = 0$ the cylinder starts to roll without slipping from rest in the accelerating train, and the weight is at the top of its arc as shown in the figure below.



(a) Determine the Lagrangian for the angle $\phi(t)$. Here $x \equiv R\phi$ is the position of the center of the cylinder relative to the back of the train (see figure). Show that the Lagrangian for ϕ may be written in a time independent form

$$L = \frac{1}{2} m_{\mathrm{eff}}(\phi)\, R^2\, \dot{\phi}^2 - U(\phi)\,, \tag{2}$$

where $m_{\mathrm{eff}}(\phi)$ and $U(\phi)$ are specific functions of ϕ

$$m_{\mathrm{eff}}(\phi) = 2m + 2m_0(1 + \cos\phi) \tag{3}$$

$$U(\phi) = (m + m_0) a_0\, R\phi + m_0 a_0 R \sin(\phi) + m_0 g \cos\phi \tag{4}$$

Hint: As in part (d) of the last problem, in the train's frame the acceleration acts like an additional gravitational field of magnitude a_0 pulling in the negative x direction.

(b) What is the speed of the cylinder after it rolls for two complete turns.

Hint: use the first integral $h(q, \dot{q}, t)$ associated with the Lagrangian in Eq. (2).

Problem 3. Legendre transform tutorial

The Legendre transform and its properties are used everywhere in all fields of physics. Recall that the Legendre transform of a concave up function $U(x)$ is[1]

$$V(s) \equiv \mathrm{max_x[sx - U(x)]} \tag{6}$$

where $\mathrm{max_x}$ denotes the maximum value of $sx - U(x)$ as x is changed. This maximum is a function of the external parameter s.

(a) Determine the Legendre transform of $\frac{1}{2}k(x - x_0)^2$

(b) (Optional. Not graded) Show that the Legendre transform of $\log(1 - e^{-x})$ with $x > 0$ is

$$V(s) = -s \log s + (1 + s) \log(1 + s) \tag{7}$$

What is the appropriate range of s? Explain.

This problem is related to the entropy of an ideal gas of bosons.

(c) Consider a potential energy function $U(x)$ (concave up) for a particle in a potential. Suppose that an external force is now applied of magnitude f_0. Relate the minimum value of the new potential $U(x, f_0)$ energy function (in the presence of f_0) to the Legendre transform of U. How could you determine the minimum value of the potential energy $U(x)$ (without the force) from $V(s)$?. If you wish you can check your result using (a).

(d) Working in the setup of the previous problem, show that the x-coordinate of the minimum of $U(x, f_0)$ with and without the external force f_0 can be determined entirely from $V(s)$? Check your general expression using the result of (a).

(e) Show that

$$\frac{\partial^2 V(s)}{\partial s^2} \frac{\partial^2 U}{\partial x^2} = 1 \tag{8}$$

The Legendre transform of a potential with several coordinates, say x^1 and x^2 for example,

$$V(s_1, s_2) \equiv s_1 x^1 + s_2 x^2 - U(x^1, x^2) \tag{9}$$

In this case Eq. (8) becomes a matrix equation. Defining

$$V^{ij} \equiv \frac{\partial^2 V(s)}{\partial s_i \partial s_j} \tag{10}$$

$$U_{ij} \equiv \frac{\partial^2 U(s)}{\partial x^i \partial x^j} \tag{11}$$

[1] If the function where concave down one would look for a minimum instead of a maximum. So it probably would be better to define

$$V(s) = \mathrm{extrm}_x[sx - U(x)] \tag{5}$$

where extrm_x denotes the extremum of a function as x is changed. Sometimes $V(s)$ is defined with a relative minus to what is done here, e.g. $V(s) = \min_x[U(x) - sx]$. It may be (marginally more) helpful to adopt this definition in part (c) below.

the generalization of Eq. (8) reads[2]

$$V^{ij}U_{j\ell} = \delta^i_{\ \ell} \tag{12}$$

i.e. U and V are inverse matrices.

For the Lagrangian a discussed in class

$$L = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j + b_i\dot{q}^j - U(q) \tag{13}$$

Write down the Hamiltonian for this Lagrangian (derived in class) and describe how the theorem in Eq. (12), is corroborated by its form.

[2]The proof is a straightforward generalization of the 1d case.