Problem 1. Equivalent Lagrangians

(a) (Goldstein) Let $L(q, \dot{q}, t)$ be the Lagrangian for a particle with coordinate q, which satisfies the Euler-Lagrange equations. Show that the Lagrangian

$$L' = L + \frac{dF(q,t)}{dt} \tag{1}$$

yields the same Euler-Lagrange equations as L where F is an arbitrary differentiable function. Give a proof based on and the action principle. We say that L and L' are equivalent. (If you feel like it you might also like to check directly that the EOM are the same.)

(b) (Goldstein) Using the previous problem (Problem 3), what is the equation of motion resulting from

$$L = -\frac{1}{2}mq\ddot{q} - \frac{1}{2}m\omega_0^2 q^2 \tag{2}$$

and what is it related to? Explain why this equation of motion is obvious from the Lagrangian in Eq. (2) and the result of part (a).

(c) Consider the action of a free particle

$$S[\boldsymbol{r}(t)] = \int \mathrm{d}t\, C\boldsymbol{v}^2 \tag{3}$$

where $C = m/2$ is a constant normally associated with the mass. Show that the action is unchanged (up to boundary terms) by a Galilean transformation, and hence the transformed version gives the same EOM. If the Lagrangian took the form $L = Cv^4$ this would not have been the case. Thus requiring Gallilean invariance fixes the form the velocity dependent action to involve the kinetic energy.

(d) Consider a fricitionless block of mass m in one dimension. The block sits on a train, which accelerates with constant acceleration a_0. The block experiences no forces, and thus the action of the block is simply the free one

$$S = \int \mathrm{d}t \frac{1}{2} m v_g^2\,, \tag{4}$$

where $v_g(t)$ is the velocity relative to the ground. Let $v(t)$ denote the velocity of the block relative to the back of the train.

(i) Write down the relation between $v(t)$ and $v_g(t)$, and substitute into Eq. (4) to determine the Lagranngian for $v(t)$.

(ii) Show that this Lagrangian is equivalent to that of a particle in a potential $U(x) = ma_0 x$ where x is the position of the particle relative to the back of the train, and interpret the result.

Solution:

(a) Clearly the the action with L' only differs by an endpoint contribution

$$S'[q] = \int_{t_1}^{t_2} dt\,(L + \frac{dF}{dt}) \tag{5}$$

$$= F(q(t),t)|_{t_1}^{t_2} + \int_{t_1}^{t_2} dt\,L \tag{6}$$

$$= F(q(t),t)|_{t_1}^{t_2} + S[q] \tag{7}$$

Since $F(q,t)$ is only on the boundary, it does not change as we vary q. Thus the variation of the action $S'[q]$ (with the endpoints fixed) is the same the variation of $S[q]$ (with endpoints fixed) yielding the same equations of motion

$$\delta S'[q, \delta q] = \delta S[q, \delta q] = 0 \tag{8}$$

(b) We can use the equations of motion for the previous problem

$$-\frac{1}{2}m\ddot{q} + 0 + \left(-\frac{1}{2}m\ddot{q} - m\omega_0^2 q\right) = 0 \tag{9}$$

i.e. the equation of motion of the SHO

$$m\ddot{q} + m\omega_0^2 q = 0 \tag{10}$$

This is obvious, as we can integrate by parts, leaving the SHO oscillator lagrangian plus neglectable total derivs:

$$L = -\frac{1}{2}mq\ddot{q} - \frac{1}{2}m\omega_0^2 q^2 \tag{11}$$

$$= \frac{1}{2}m\dot{q}^2 - \frac{1}{2}m\omega_0^2 q^2 - \frac{d}{dt}\left(\frac{1}{2}mq\dot{q}\right) \tag{12}$$

$$= L_{\rm SHO} + \frac{d}{dt}\,(\text{ignore}-\text{me}) \tag{13}$$

(c) The action for coordinate $\boldsymbol{r}$ is

$$S[\boldsymbol{r}] = \int dt C \dot{\boldsymbol{r}}^2 \tag{14}$$

In a frame moving with velocity $\boldsymbol{u}$ to right, the coordinates in the new frame $\boldsymbol{r}'$ are related to the old coordinate $\boldsymbol{r}$ via

$$\dot{\boldsymbol{r}} \to \boldsymbol{r}' = \boldsymbol{r} - \boldsymbol{u}\,t \tag{15}$$

or $\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{u}\,t$. The action for the coordinate $\boldsymbol{r}'$ is therefore

$$S[\boldsymbol{r}'] = \int dt C \dot{\boldsymbol{r}}^2 \tag{16}$$

$$= \int dt C \dot{\boldsymbol{r}}'^2 + 2C\dot{\boldsymbol{r}}' \cdot \boldsymbol{u} + Cu^2 \tag{17}$$

$$= \int dt C \dot{\boldsymbol{r}}'^2 + \frac{d}{dt}\left(2C\boldsymbol{r}' \cdot \boldsymbol{u} + Cu^2 t\right) \tag{18}$$

$$= \int dt C \dot{\boldsymbol{r}}'^2 + \text{total derivs} \tag{19}$$

Thus the equation of motion for $\boldsymbol{r}'$ will be the same as for $\boldsymbol{r}$ since they have the same action up to a boundary term.

(d) The position of the back of the train is $\frac{1}{2}a_0 t^2$ and thus the coordinates are related

$$x_g(t) = x(t) + \tfrac{1}{2}a_0 t^2 \,. \tag{20}$$

So

$$\dot{x}_g = \dot{x} + a_0 t \tag{21}$$

We have

$$S[x] = \int \frac{1}{2}m(\dot{x} + a_0 t)^2 \tag{22}$$

$$= \int \frac{1}{2}m\dot{x}^2 + m\dot{x}a_0 t + \frac{1}{2}ma_0^2 t^2 \tag{23}$$

The last term is a independent of x and can be ignored (we only care about the variation of δS and this term is constant under the variation). The second to last term can be integrated by parts

$$S[x] = \int \frac{1}{2}m\dot{x}^2 + m\dot{x}a_0 t \tag{24}$$

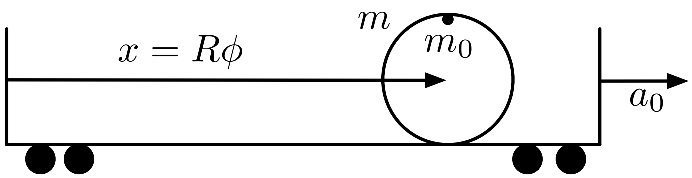
$$= \int \frac{1}{2}m\dot{x}^2 - ma_0 x + \frac{d}{dt}(mxa_0 t) \tag{25}$$

$$= \int \frac{1}{2}m\dot{x}^2 - ma_0 x + (\text{total derivs}) \tag{26}$$

This (without the irrelevant total-derivs) is the action for a particle experiencing a constant force $F_x = -ma_0$, i.e. a force pushing the particle to the left, towards the back of the train. This is the expected effective force because we are viewing the motion in the coordinate system of the accelerating train.

Problem 2. A cylinder on a train

Consider a cylinder-like contraption consisting of a cylindrical ring of mass m and radius R, and a small weight of mass m_0 fixed to the rim of the ring (see below). At time $t = 0$ the cylinder starts to roll without slipping from rest in the accelerating train, and the weight is at the top of its arc as shown in the figure below.



(a) Determine the Lagrangian for the angle $\phi(t)$. Here $x \equiv R\phi$ is the position of the center of the cylinder relative to the back of the train (see figure). Show that the Lagrangian for ϕ may be written in a time independent form

$$L = \frac{1}{2} m_{\text{eff}}(\phi)\, R^2\, \dot{\phi}^2 - U(\phi)\,, \tag{27}$$

where $m_{\text{eff}}(\phi)$ and $U(\phi)$ are specific functions of ϕ

$$m_{\text{eff}}(\phi) = 2m + 2m_0(1 + \cos\phi) \tag{28}$$

$$U(\phi) = (m + m_0)a_0\, R\phi + m_0 a_0 R \sin(\phi) + m_0 g R \cos\phi \tag{29}$$

Hint: As in part (d) of the last problem, in the train's frame the acceleration acts like an additional gravitational field of magnitude a_0 pulling in the negative x direction.

(b) What is the speed of the cylinder after it rolls for two complete turns.

Hint: use the first integral $h(q, \dot{q}, t)$ associated with the Lagrangian in Eq. (27).

Solution:

(a) In the train's frame the acceleration acts like an additional gravitational potential energy $U = ma_0 x$, with a force $F^x = -ma_0$. The Lagrangian of the contraption is thus

$$L = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}I\dot{\phi}^2 + \frac{1}{2}m_0(\dot{x}_1^2 + \dot{y}_1^2) - ma_0x - m_0a_0x_1 - m_0gy_1 \tag{30}$$

where $x = R\phi$. The coordinates of the weight are

$$x_1 = R\phi + R\sin\phi \tag{31}$$
$$y_1 = R\cos\phi \tag{32}$$

So with $I = mR^2$ and $x = R\phi$ we have after minor algebra

$$L = \frac{1}{2}m_{\text{eff}}(\phi)R^2\dot{\phi}^2 - U(\phi) \tag{33}$$

where

$$m_{\text{eff}}(x) = 2m + 2m_0(1 + \cos\phi) \tag{34}$$
$$U = ma_0x + m_0a_0\left(x + R\sin\phi\right) + m_0gR\cos\phi \tag{35}$$

Here it is understood that $x = R\phi$.

(b) The system has a first integral and therefore

$$E = \frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x) \tag{36}$$

is constant, where it is understood that $x = R\phi$. Thus

$$\left.\frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x)\right|_{\text{final}} = \left.\frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x)\right|_{\text{initial}} \tag{37}$$

Since the initial kinetic energy is zero and the initial potential energy is m_0gR. Thus for a given x we have

$$\frac{1}{2}m_{\text{eff}}(x)\dot{x}^2 + U(x) = m_0gR \tag{38}$$

After two turns $x = -2(2\pi R)$ and $\phi = 4\pi$. Thus $U(x) = -(m + m_0)a_0(4\pi R) + m_0gR$, while the mass is

$$m_{\text{eff}}(x_{\text{final}}) = 2m + 4m_0 \tag{39}$$

So we find

$$\dot{x} = \sqrt{\frac{4\pi a_0(m + m_0)R}{2m + 4m_0}} \tag{40}$$

Problem 3. Legendre transform tutorial

The Legendre transform and its properties are used everywhere in all fields of physics. Recall that the Legendre transform of a concave up function $U(x)$ is[1]

$$V(s) \equiv \mathrm{max_x[sx - U(x)]} \tag{42}$$

where $\mathrm{max_x}$ denotes the maximum value of $sx - U(x)$ as x is changed. This maximum is a function of the external parameter s.

(a) Determine the Legendre transform of $\frac{1}{2}k(x - x_0)^2$

(b) (Optional. Not graded) Show that the Legendre transform of $\log(1 - e^{-x})$ with $x > 0$ is

$$V(s) = -s \log s + (1 + s) \log(1 + s) \tag{43}$$

What is the appropriate range of s? Explain.

This problem is related to the entropy of an ideal gas of bosons.

(c) Consider a potential energy function $U(x)$ (concave up) for a particle in a potential. Suppose that an external force is now applied of magnitude f_0. Relate the minimum value of the new potential $U(x, f_0)$ energy function (in the presence of f_0) to the Legendre transform of U. How could you determine the minimum value of the potential energy $U(x)$ (without the force) from $V(s)$?. If you wish you can check your result using (a).

(d) Working in the setup of the previous problem, show that the x-coordinate of the minimum of $U(x, f_0)$ with and without the external force f_0 can be determined entirely from $V(s)$? Check your general expression using the result of (a).

(e) Show that

$$\frac{\partial^2 V(s)}{\partial s^2}\frac{\partial^2 U}{\partial x^2} = 1 \tag{44}$$

The Legendre transform of a potential with several coordinates, say x^1 and x^2 for example,

$$V(s_1, s_2) \equiv s_1 x^1 + s_2 x^2 - U(x^1, x^2) \tag{45}$$

In this case Eq. (44) becomes a matrix equation. Defining

$$V^{ij} \equiv \frac{\partial^2 V(s)}{\partial s_i \partial s_j} \tag{46}$$

$$U_{ij} \equiv \frac{\partial^2 U(s)}{\partial x^i \partial x^j} \tag{47}$$

[1] If the function where concave down one would look for a minimum instead of a maximum. So it probably would be better to define

$$V(s) = \mathrm{extrm}_x[sx - U(x)] \tag{41}$$

where extrm_x denotes the extremum of a function as x is changed. Sometimes $V(s)$ is defined with a relative minus to what is done here, e.g. $V(s) = \min_x[U(x) - sx]$. It may be (marginally more) helpful to adopt this definition in part (c) below.

the generalization of Eq. (44) reads[2]

$$V^{ij}U_{j\ell} = \delta^i_{\ \ell} \tag{48}$$

i.e. U and V are inverse matrices.

For the Lagrangian a discussed in class

$$L = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j + b_i\dot{q}^j - U(q) \tag{49}$$

Write down the Hamiltonian for this Lagrangian (derived in class) and describe how the theorem in Eq. (48), is corroborated by its form.

[2]The proof is a straightforward generalization of the 1d case.

Legendre Transform Tutorial

(a) Since $u(x) = \frac{1}{2} k (x - x_0)^2$

Then

$$V(s) = \underset{x}{\text{extrm}} \{ sx - u(x) \}$$

So the extremal point is when $d/dx(sx - u) = 0$:

$$s = u'(x) = k(x - x_0) \Rightarrow x = \frac{s + kx_0}{k}$$

So the extemal value is

$$V(s) = s \left(\frac{s + kx_0}{k} \right) - \frac{1}{2} k \left(\frac{s + kx_0}{k} - x_0 \right)^2$$

$$\boxed{V(s) = \frac{(s + x_0)^2}{2k} - \frac{x_0^2}{2k}}$$

(b) Following general procedure we solve

$$s = \frac{\partial u(x)}{\partial x} \quad \text{for} \quad x(s)$$

And

$$V(s) = s\, x(s) - u(x(s))$$

Here

$$s = \frac{e^{-x}}{1-e^{-x}} \qquad x = \log\frac{(1+s)}{s}$$

Substituting into $V(s) = s\,x(s) - U(x(s))$ yields

$$\boxed{V(s) = (1+s)\log(1+s) - s\log s}$$

c) The work done by the external force f_0 is $f x_0$. So the change in potential is $\Delta U = -f x_0$. So the full potential is

$$\hat{U}(x, f_0) = U(x) - f_0 x$$

The minimum of $\hat{U}$ is found

$$\min_x (U(x) - f_0 x) = -V(f_0)$$

Then clearly

$$\boxed{\min_x (U(x)) = -V(f_0)\Big|_{f_0 = 0}}$$

Note: from a) $V(0) = \frac{y^2}{2k} - \frac{y^2}{2k} = 0$

Which clearly is the minmum of

$$\frac{1}{2} k (x - x_0)^2$$

(d) Now first we establish a familiar fact about LT's:

$$x(f_0) = \frac{dV}{df_0}$$

Prf

$$dU = f_0 \, dx$$

Here f is the applied force to have a steady position x

$$d(U - f_0 x) = -x \, df_0$$

- So since $V = f_0 x - U$, $dV/df_0 = x(f_0)$.
- Now we wan't x at the minimum, where the applied force is zero. So x_{min} is

$$\boxed{x_{min} = \left.\frac{dV}{df}\right|_{f=0}}$$

- So from part (a):

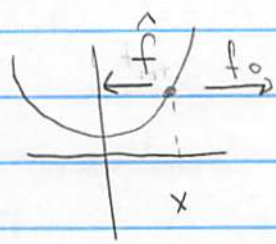
this is x_{min}

$$\left.\frac{dV}{df}\right|_{f=0} = \left.\frac{d}{df}\left[\frac{(f + x_0)^2}{2K} - \frac{x_0^2}{2K}\right]\right|_{f=0} = x_0 \text{, as claimed.}$$

(e) Now we estabish

$$\frac{\partial^2 V}{\partial f^2} \frac{\partial^2 U}{\partial x^2} = 1$$

- So note $f_0 = \dfrac{\partial U}{\partial x}$. This is the force that must be applied to attain a steady state at point x



$\hat{f} = -\dfrac{\partial U}{\partial x}$ is the force the system produces

$f_0 \equiv$ is the force we apply to attain x as a steady state.

- The previous item Showed

$$x(f_0) = \frac{\partial V(f_0)}{\partial f_0}$$

- Now

$$\frac{dx}{df_0} \frac{df_0}{dx} = 1$$

So

$$\boxed{\frac{d^2V}{df_0^2}\,\frac{d^2u}{dx^2} = 1}$$

(e) $L = \frac{1}{2}\, a_{ij}\, \dot{q}^i \dot{q}^j + b_i \dot{q}^i - U(q)$

So the Hamiltonian is:

$$H = \frac{(a^{-1})^{ij}}{2}(p_i - b_i)(p_j - b_j) + U(q)$$

Then accordingly:

$$\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j} = a_{ij}$$

While H is the LT of L:

$$\frac{\partial H}{\partial p^i \partial p^j} = (a^{-1})^{ij}$$

Then

$$\boxed{\frac{\partial L}{\partial \dot{q}^i \partial \dot{q}^j}\,\frac{\partial^2 H}{\partial p_j \partial p_k} = \delta_i^k}$$

which corroborates the theorem