Problem 1. Particle in an electro-magnetic field

A non-relativistic particle of charge q in a electro-magnetic field is described by the Lagrangian (try to remember this!)

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - q\phi + q\frac{\dot{\boldsymbol{r}}}{c} \cdot \boldsymbol{A} \tag{1}$$

where $\phi(t, \boldsymbol{r}(t))$ is the scalar potential, and $\boldsymbol{A}(t, \boldsymbol{r}(t))$ is the vector potential of electricity and magnetisim. The electric and mangic fields are related to ϕ and $\boldsymbol{A}$ through

$$\boldsymbol{E}(t, \boldsymbol{r}) = -\nabla\phi - \frac{1}{c}\partial_t \boldsymbol{A} \qquad E_i = -\partial_i \phi - \frac{1}{c}\partial_t A_i \tag{2}$$

$$\boldsymbol{B}(t, \boldsymbol{r}) = \nabla \times \boldsymbol{A} \qquad \epsilon_{ijk} B^k = \partial_i A_j - \partial_j A_i \tag{3}$$

(a) Show that the Euler-Lagrange equations give the expected EOM for a particle experiencing the force law: $\boldsymbol{F} = q(\boldsymbol{E} + \frac{\boldsymbol{v}}{c} \times \boldsymbol{B})$.

(b) Compute the canonical momementa $\boldsymbol{p}$. How is this related to the so called kinetic momentum $\boldsymbol{p}_{\text{kin}} = m\dot{\boldsymbol{r}}$? Use part (a) to determine

$$\frac{d(\boldsymbol{p} - \frac{q}{c}\boldsymbol{A})}{dt} \tag{4}$$

(c) Determine the Hamiltonian $H(\boldsymbol{r}, \boldsymbol{p})$ and Hamiltonian function $h(\boldsymbol{r}, \dot{\boldsymbol{r}})$.

$H(\boldsymbol{r}, \boldsymbol{p})$ and $h(\boldsymbol{r}, \dot{\boldsymbol{r}})$ return the same value (at corresponding points), but have different functional forms (meaning that they have different dependences on the arguements). A mathematician would (correctly) say that they are different functions, but we (too) loosely say that they are the "same".

(d) (Optional. Not graded) Compute $d\boldsymbol{p}/dt$ from the Hamiltonian formalism, and show that it leads to the somewhat unintuitive result

$$\frac{dp_i}{dt} = -q\partial_i\phi + \frac{q}{c}\,\dot{r}^j \partial_i A_j \tag{5}$$

Rederive Eq. (4) from Hamilton's equations of motion.

Solution:

(a) Constructing the Euler Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^i}\right) = \frac{\partial L}{\partial r^i} \tag{6}$$

$$\frac{d}{dt}\left(m\dot{r}_i + \frac{q}{c}A_i\right) = - q\partial_i\phi + \frac{q}{c}\dot{r}^j\,\partial_i A_j \tag{7}$$

where we used the indexed notation, e.g.

$$\partial_i A_j = \frac{\partial}{\partial r^i}A_j(t, \boldsymbol{r}) \tag{8}$$

Then differentiating away we have

$$\frac{d}{dt}A_i = \partial_t A_i + \partial_j A_i\,\dot{r}^j \tag{9}$$

So

$$\frac{d}{dt}(m\dot{r}_i) = q\left(-\partial_i\phi - \frac{1}{c}\partial_t A_i\right) + \frac{q}{c}v^j\left(\partial_i A_j - \partial_j A_i\right) \tag{10}$$

Recognizing the electric and magnetic fields

$$\partial_i A_j - \partial_j A_i = \epsilon_{ijk}B_k \tag{11}$$

we find

$$\frac{d}{dt}(m\dot{r}_i) = qE_i + \frac{q}{c}\epsilon_{ijk}v^j B^k \tag{12}$$

which is the Lorentz force law $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v}/c \times \boldsymbol{B})$

(b) The canonical momentum is

$$p_i = \frac{\partial L}{\partial \dot{r}^i} = m\dot{r}_i + \frac{q}{c}A_i \tag{13}$$

We have $\boldsymbol{p}_i = \boldsymbol{p}_{\rm kin} + q/c\boldsymbol{A}$. So

$$\frac{d(\boldsymbol{p} - q/c\boldsymbol{A})}{dt} = \frac{d\boldsymbol{p}_{\rm kin}}{dt} = \boldsymbol{F} \tag{14}$$

(c) The Lagrangian is cast in a general form discussed in class

$$L = \frac{1}{2}m\,\delta_{ij}\,\dot{r}^i\,\dot{r}^j + \frac{q}{c}\dot{r}^i A_i - q\phi(r) \tag{15}$$

We find from general results derived in class

$$h = \frac{1}{2} m \, \delta_{ij} \, \dot{r}^i \dot{r}^j + q\phi \tag{16}$$

The canonical momentum

$$p_i = m\dot{r}_i + \frac{q}{c} A_i \tag{17}$$

And then

$$H(\boldsymbol{p}, \boldsymbol{r}) = \frac{1}{2m} \delta^{ij} (p_i - \frac{q}{c} A_i)(p_j - \frac{q}{c} A_j) + q\phi \tag{18}$$

$$H(\boldsymbol{p}, \boldsymbol{r}) = \frac{1}{2m} (\boldsymbol{p} - \frac{q}{c} \boldsymbol{A})^2 + q\phi \tag{19}$$

(d) The Hamilton equations of motion give

$$\frac{dr^i}{dt} = \frac{p^i - q/cA^i}{m} \tag{20}$$

$$\frac{dp_i}{dt} = -q\partial_i \phi + \frac{q}{c} \left(\frac{p^j - q/cA^j}{m} \right) \partial_i A_j \tag{21}$$

$$= -q\partial_i \phi - \frac{q}{c} \dot{r}^j \, \partial_i A_j \tag{22}$$

In order to see the force law we subtract:

$$\frac{q}{c} \frac{dA_i}{dt} = \frac{q}{c} \partial_t A_i + \frac{q}{c} \partial_j A_i \dot{r}^j \tag{23}$$

which gives

$$\frac{d}{dt} (p_i - \frac{q}{c} A_i) = q(\boldsymbol{E} + \dot{\boldsymbol{r}}/c \times \boldsymbol{B})_i \tag{24}$$

Problem 2. A Routhian tutorial and the effective potential

Consider the Kepler Lagrangian again:

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r) \tag{25}$$

There are two variables r and ϕ with associated momenta p_r and p_ϕ. The Hamiltonian is is formed by Legendre transforming with respect to r and ϕ

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L(r, \dot{r}, \phi, \dot{\phi})\,. \tag{26}$$

It can be convenient to Legendre transform with respect to only some of the variables instead of all of them. We define the *Routhian*[1]:

$$R(r, \dot{r}, \phi, p_\phi) \equiv p_\phi\dot{\phi} - L(r, \dot{r}, \phi, \dot{\phi})\,, \tag{27}$$

which serves as a Hamiltonian for ϕ, but a Lagrangian for r. This is especially helpful when some of the coordinates are cyclic (ϕ in this case). The p_ϕ are then just constants (both in the equation of motion *and* in the action), and we have effectively a Lagrangian for the remaining (non-cyclic) coordinates.

(a) From the Lagrange equations of motion, show that the Routhian equations of motion (for a generic Lagrangian not just Eq. (25)) are

$$\frac{d\phi}{dt} = \frac{\partial R}{\partial p_\phi} \tag{28}$$

$$\frac{dp_\phi}{dt} = -\frac{\partial R}{\partial \phi} \tag{29}$$

$$\frac{d}{dt}\frac{\partial R}{\partial \dot{r}} = \frac{\partial R}{\partial r} \tag{30}$$

(b) Determine $R(r, \dot{r}, \phi, p_\phi)$ for the Lagrangian in Eq. (25) and the Routhian equations of motion. You should find[2]

$$-R = \frac{1}{2}m\dot{r}^2 - V_{\text{eff}}(r, p_\phi) \tag{31}$$

where $V_{\text{eff}}(r, p_\phi)$ was defined in class and the equation of motions are

$$m\ddot{r} = -\frac{\partial V_{\text{eff}}(r, p_\phi)}{\partial r} \tag{32}$$

$$p_\phi = \text{const} \tag{33}$$

Now might be a good time to review the appropriate comments on bottom of pg.2 and 3 in lecture to appreciate the how the Routhian can help, i.e. we want $(\partial V_{\text{eff}}/\partial r)_{p_\phi}$.

[1] Edward John Routh was a physicist of some repute. He was also an outstanding educator at Cambridge.

[2] Note that the sign of R is conventional. The choice here is nice in that the Hamiltonian part of the equations (Eq. (28) and Eq. (29)) takes the form of Hamilton's equations. But then, R is minus the effective Lagrangian for the non-cyclic coordinates. We will get around this "difficulty" by presenting $-R$.

(c) A particle of mass m is confined to move on the surface of a sphere. It moves freely on the surface but experiences the acceleration of gravity g:

(i) Write down the Lagrangian for this system using the spherical angular variables θ, ϕ.

(ii) Form the Routhian for this system by Legendre transforming with respect to the cyclic coordinate.

(iii) Sketch the effective potential of θ for p_ϕ small and large, after defining what large and small means. Determine the stationary point of θ at large p_ϕ, and briefly describe the result physically.

Routhian tutorial

a) $$dL = p_\phi \, d\dot{\phi} + \frac{\partial L}{\partial \phi} d\phi + \text{other spectators}$$

Then

- $$dR = d(p_\phi \dot{\phi} - L)$$

$$= \dot{\phi} \, dp_\phi + p_\phi \, d\dot{\phi} - dL$$

$$= \dot{\phi} \, dp_\phi - \frac{\partial L}{\partial \phi} d\phi - \text{other spectators}$$

- So $$\boxed{\frac{\partial R}{\partial p_\phi} = \dot{\phi}}$$

and $$\frac{\partial R}{\partial \phi} = -\frac{\partial L}{\partial \phi}$$

- Now the Lagrange EOM for ϕ are

$$\frac{d}{dt} p_\phi = \frac{\partial L}{\partial \phi} \Rightarrow \boxed{\frac{dp_\phi}{dt} = -\frac{\partial R}{\partial \phi}}$$

- The remaining variables are spectators

$$\frac{\partial R}{\partial \dot{r}} = -\frac{\partial L}{\partial \dot{r}} \qquad \frac{\partial R}{\partial r} = -\frac{\partial L}{\partial r}$$

So from the Euler Lagrange EOM read

$$-\frac{d}{dt}\frac{\partial R}{\partial \dot{r}} = -\frac{\partial R}{\partial r}$$

b) Now

$$L = \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 - U(r)$$

- Then

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2\dot{\phi} \implies \dot{\phi} = \frac{p_\phi}{mr^2}$$

- So

$$-R = L - p_\phi\dot{\phi} = \left(\frac{1}{2}m\dot{r}^2 + \frac{p_\phi^2}{2mr^2} - U(r)\right) - \frac{p_\phi^2}{mr^2}$$

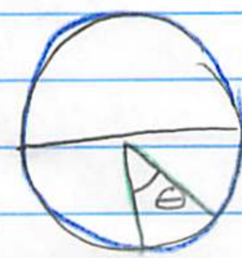
$$-R = \frac{1}{2}m\dot{r}^2 - \left(\frac{p_\phi^2}{2mr^2} + U(r)\right)$$

(c) Then first we need coordinates

$$x = a \sin\theta \cos\phi$$

$$y = a \sin\theta \sin\phi$$

$$z = -a\cos\theta$$



- So straightforward algebra or geometry gives

$$T = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\dot{y}^2 + \frac{1}{2} m\dot{z}^2$$

$$= \frac{1}{2} m a^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2)$$

And since $U = +mgz$ and $L = T - U$ we have

$$(1) \quad L = \frac{1}{2} m a^2 (\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2) + mga\cos\theta$$

(ii) The ϕ coordinate is cyclic

$p_\phi = ma^2 \sin^2\theta\, \dot{\phi}$ = angular momentum in the z direction

The Routhian is

$$-R = L - p_\phi \dot{\phi}$$

$$-R = \frac{1}{2} m a^2 \dot{\theta}^2 - \underbrace{\left(\frac{p_\phi^2}{2ma^2 \sin^2\theta} - mga\cos\theta \right)}_{V_{eff}}$$

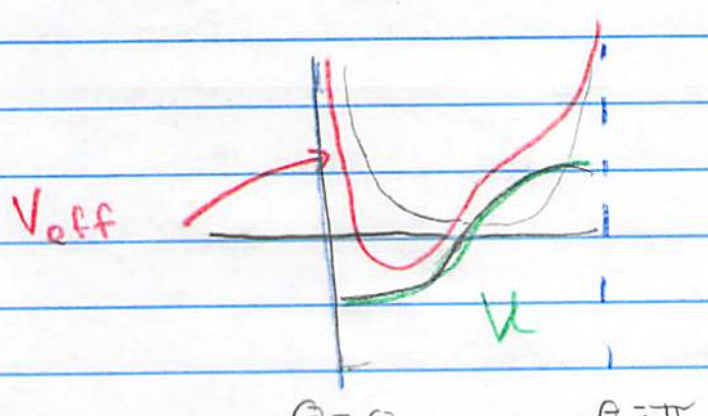
(iii) Then

$$V_{eff} = \frac{p_\phi^2}{2ma^2\sin^2\theta} - mga\cos\theta$$

So p_ϕ is small when

$$\frac{p_\phi^2}{2ma^2} \ll mga$$

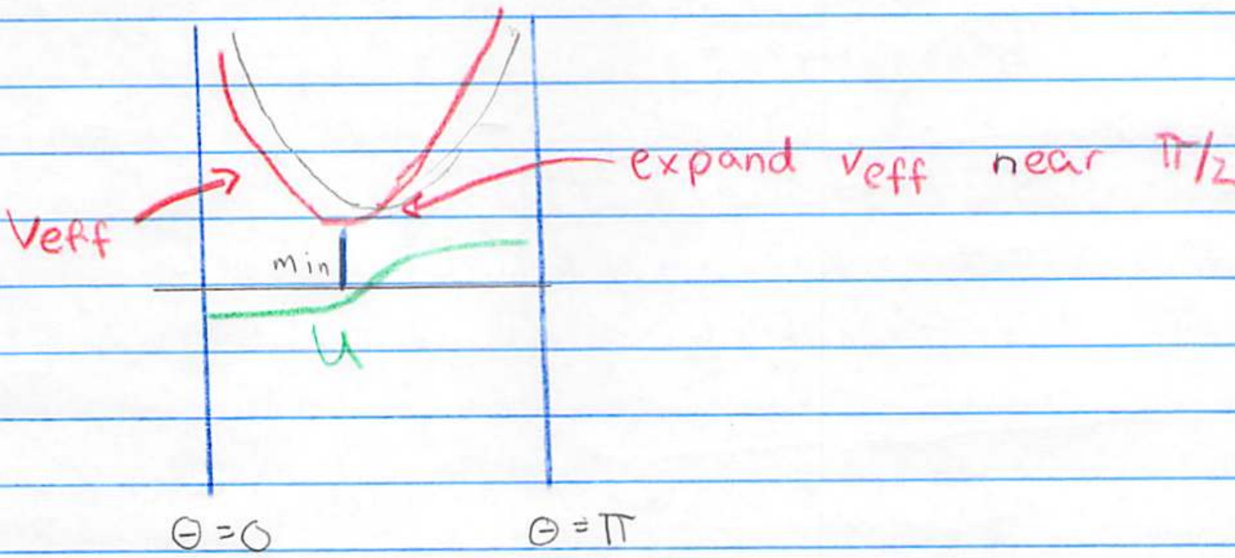
and large when the equality is reversed



At

- Small p_ϕ, the potential is only modified at $\theta = 0, \pi$

- For Large p_ϕ the potential scarcely modifies the centrifugal barrier



- The minimum of $V(\Theta)$ determines the stationary point. Near $\Theta = \pi/2$ we expand $\delta = \Theta - \pi/2$

$$\frac{1}{\sin^2\Theta} \simeq \frac{1}{(1-\delta^2/2)^2} \simeq 1 + \delta^2$$

$$-\cos\Theta \simeq \delta$$

So near $\Theta = \pi/2$:

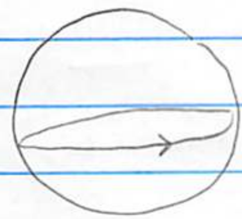
- $V_{eff} = \frac{p^2_\phi}{2ma^2}(1+\delta^2) + mga\delta$, with minimum when

- $\frac{\partial V_{eff}}{\partial \delta} = \frac{p_\phi^2}{ma^2}\delta + mga = 0$

- So the stationary point is just below $\pi/2$

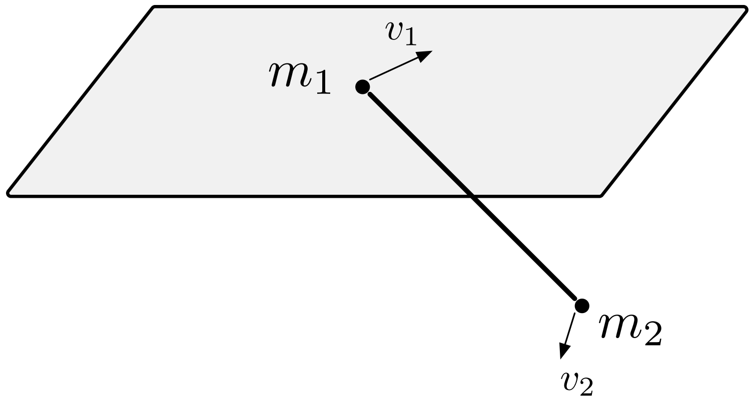
$$\delta = -mg\left(\frac{ma^3}{p_\phi^2}\right)$$

- So in the limit $p_\phi \to \infty$ the bead is going around so fast it is pushed by the centrifugal force to $\theta \simeq \pi/2$. But, then gravity sets in decreasing the angle slightly. The deficit from $\pi/2$ is the ratio of the gravitational force to the (Large) centrifugal force.



Problem 3. A sliding conical pendulum

Consider two beads connected by a light rod of length ℓ. The first bead has mass m_1 and is constrained to lie in the x, y plane, but may move freely in this plane. The second bead has mass m_2 and can move freely in all three dimensions, and can pass freely through the x, y plane. The system sits in the earths gravitational field $\boldsymbol{g} = -g\,\hat{\mathbf{z}}$.



(a) Determine the distance from m_1 to the center of mass. You should find

$$\ell_{\rm cm} = \alpha\ell, \qquad \alpha \equiv \frac{m_2}{M}, \qquad M \equiv (m_1 + m_2)\,, \tag{34}$$

which establishes some notation used below.

(b) Clearly define some appropriate generalized coordinates for the system, and write down the Lagrangian of the system in terms of these coordinates.

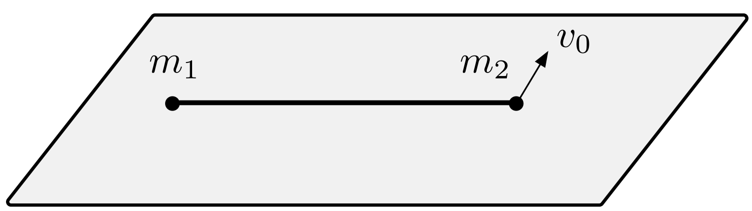
Hint: The cartesian coordinates (X, Y, Z) of the center of mass is an excellent choice. Then I used the the spherical coordinates θ and ϕ to orient the rod relative to the center of mass. I find the Lagrangian takes the form

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_0(\theta)\ell^2\dot{\theta}^2 + +\frac{1}{2}\mu\ell^2\sin^2\theta\dot{\phi}^2 + Mg\alpha\ell\cos\theta \tag{35}$$

where $m_0 = M\alpha^2\sin^2\theta + \mu$ and $\mu = m_1m_2/M$ is the reduced mass.

(c) Identify all integrals of the motion.

Now consider the case where the first bead is initially at rest and the second bead initially has velocity v_0 in the x, y plane, and perpendicular to the rod, before beginning to fall (see below).



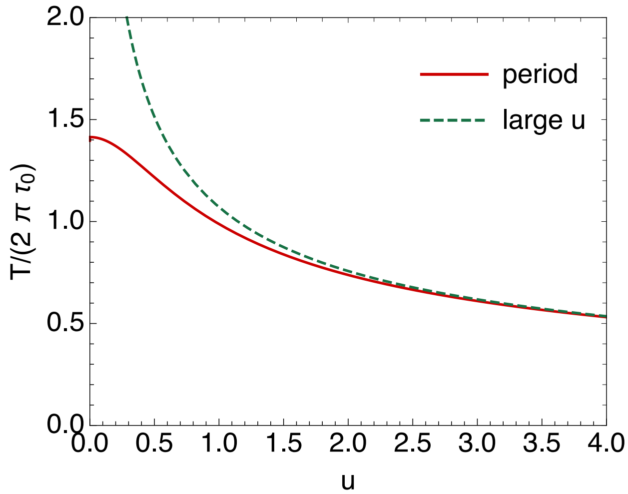


Figure 1: The period of the motion (normalized by $2\pi\ell/v_0$) as a function of u (see text).

(d) Describe qualitatively the subsequent motion of the system. In what Galilean frame is the motion periodic? Explain.

(e) (i) The pendulum swings down from an initial angle of $\pi/2$ relative to the vertical to a minimum angle. Determine this minimum angle.

You should find

$$\cos\theta_- = \frac{-1+\sqrt{1+4u^2}}{2u} \qquad\qquad \theta_- < \pi/2\,. \tag{36}$$

where $u = Mg\alpha\ell/\frac{1}{2}\mu v_0^2$.

(ii) Determine the associated period of the motion as a definite integral. Define what is meant by large and small v_0 and describe the motion qualitatively in these two limits.

You should show that this period takes the form

$$\mathcal{T} = \tau_0\, f(u, m_1/m_2) \tag{37}$$

where $\tau_0 \equiv \ell/v_0$ and $f(u,r)$ is a dimensionless function of u and the ratio of masses $r = m_1/m_2$. Use mathematica to plot to make a plot of $\mathcal{T}/(2\pi\tau_0)$ for $m_1 = m_2$, which is exhibited above.

Solution

(a) From the picture, the center of mass is a distance $m_2\ell/M \equiv \alpha\ell$ from the first particle m_1 which is attached to the plane.

(b) It makes sense to use center of mass coordinates. Let us denote $M = m_1 + m_2$ as the total mass. The center of mass is mass coordinate $\boldsymbol{R} = (X, Y, Z)$

$$\boldsymbol{R} = \frac{m_1}{M}\boldsymbol{r}_1 + \frac{m_2}{M}\boldsymbol{r}_2 \,. \tag{38}$$

The relative coordinate is $\boldsymbol{r} = (x, y, z)$ is

$$\boldsymbol{r} = \boldsymbol{r}_1 - \boldsymbol{r}_2 \,, \tag{39}$$

and the reduced mass is $\mu = m_1 m_2/(m_1 + m_2)$. The kinetic energy is

$$T = \frac{1}{2}M\dot{\boldsymbol{R}}^2 + \frac{1}{2}\mu\dot{\boldsymbol{r}}^2 \,. \tag{40}$$

The vector $\boldsymbol{r}$ has a fixed length and is conveniently parameterized by two angles

$$x = \ell\sin(\theta)\cos\phi \,, \tag{41}$$
$$y = \ell\sin(\theta)\sin\phi \,, \tag{42}$$
$$z = \ell\cos(\theta) \,. \tag{43}$$

Now the angle θ is related to the height of the center of mass. From the picture, the center of mass is a distance $m_2\ell/M \equiv \alpha\ell$ from the first particle m_1 which is attached to the plane. We have from geometry

$$Z = -\alpha\ell\cos\theta \,. \tag{44}$$

Thus

$$\dot{Z} = \alpha\ell\sin\theta\,\dot{\theta} \,, \tag{45}$$

and then the kinetic energy is

$$T = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \alpha^2\ell^2\sin^2(\theta)\dot{\theta}^2) + \frac{1}{2}\mu\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \,. \tag{46}$$

The potential energy is $U = MgZ$. Thus the full Lagrangian is

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2 + \alpha^2\ell^2\sin^2\theta\dot{\theta}^2) + \frac{1}{2}\mu\ell^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + Mg\alpha\ell\cos\theta \,. \tag{47}$$

Since two of the terms are very similar, we define

$$m_0(\theta) = M\alpha^2\sin^2\theta + \mu \,, \tag{48}$$

leading to our final result

$$L = \frac{1}{2}M(\dot{X}^2 + \dot{Y}^2) + \frac{1}{2}m_0(\theta)\ell^2\dot{\theta}^2 + \frac{1}{2}\mu\ell^2\sin^2\theta\dot{\phi}^2 + Mg\alpha\ell\cos\theta \,. \tag{49}$$

(c) The are several cyclic coordinates owing to the symmetries of the problem.First there is the total momentum of the system

$$p_X = \frac{\partial L}{\partial \dot{X}} = M\dot{X}\,, \tag{50}$$

$$p_Y = \frac{\partial L}{\partial \dot{Y}} = M\dot{Y}\,. \tag{51}$$

Then there is the angular momentum around the Z axis.

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \mu\ell^2 \sin^2\theta\dot{\phi}\,. \tag{52}$$

Finally there is the total energy of the system

$$E = \frac{p_X^2}{2M} + \frac{p_Y^2}{2m} + \frac{1}{2}m_0(\theta)\ell^2\dot{\theta}^2 + \frac{p_\phi^2}{2\mu\ell^2\sin^2(\theta)} - Mg\alpha\ell\cos\theta\,. \tag{53}$$

(d) In a frame which moves with the center of mass the motion will be periodic. The initial momenta are $p_1^y = 0$ and $p_2^y = m_2 v_0$ and

$$p_X = 0\,, \tag{54}$$

$$p_Y = m_2 v_0\,, \tag{55}$$

and thus if we look at the motion in a frame which moves in the y-direction with velocity $v_Y = m_2 v_0/M$ the motion will be periodic.

(e) (i) The initial conditions also excites internal oscillations and orbital motion. Similarly using a bit of geometry of we have that in the center of mass frame $\dot{\phi} = v_0/\ell$ and thus ϕ angular motion is determined by the angular momentum variable

$$p_\phi = \mu\ell v_0\,. \tag{56}$$

Finally the energy is constant and is determined by the initial conditions

$$E_0 = \frac{1}{2}m_2 v_0^2 = \underbrace{\frac{1}{2}\frac{m_2^2}{M}v_0^2}_{\text{init translational KE}} + \underbrace{\frac{1}{2}\mu v_0^2}_{\text{init rotational KE}}\,. \tag{57}$$

So setting $E = E_0$, we have after minor manipulations

$$\frac{1}{2}m_0(\theta)\ell^2\dot{\theta}^2 + \frac{\mu v_0^2}{2\sin^2\theta} - Mg\alpha\ell\cos\theta = \frac{1}{2}\mu v_0^2\,. \tag{58}$$

Then we have

$$\frac{1}{2}m_0(\theta)\ell^2\dot{\theta}^2 = -\frac{1}{2}\mu v_0^2\cot^2\theta + Mg\alpha\ell\cos\theta\,. \tag{59}$$

Solving for $\dot{\theta}$

$$\frac{d\theta}{dt} = \pm\frac{v_0}{\ell}\sqrt{\frac{\mu}{m_0(\theta)}}\sqrt{-\frac{\cos^2\theta}{\sin^2\theta} + u\cos\theta}\,, \tag{60}$$

where we have defined:

$$u \equiv \frac{Mg\alpha\ell}{\frac{1}{2}\mu v_0^2} \,. \tag{61}$$

u is a dimensionless number, which is the ratio of the initial potential to initial rotational kinetic energy. τ is a timescale set by the internal energy. Since the angle is decreasing (initially) we take the negative root for the first half period.

The turning points are when $\dot{\theta}$ is zero. Solving the equation for the turning points we have

$$\cos\theta_0 = 0 \qquad \theta_+ = \pi/2 \,, \tag{62}$$

$$\cos\theta_- = \frac{-1+\sqrt{1+4u^2}}{2u} \qquad \theta_- < \pi/2 \,. \tag{63}$$

Finally there is an unphysical turning point when the pendulum has angle greater than $\pi/2$.

(ii) Integrating the equation of motion Eq. (60) we find

$$\int_0^t dt = -\frac{\ell}{v_0}\int_{\pi/2}^{\theta(t)} d\theta \sqrt{\frac{1+r\sin^2\theta}{-\cot^2(\theta)+u\cos(\theta)}} \,, \tag{64}$$

with $r \equiv m_2/m_1$. (N.B. the problem defined $r = m_1/m_2$). Here we have recognized that

$$\frac{m_0(\theta)}{\mu} = \frac{1}{\mu}\left(M\alpha^2\sin^2(\theta)+\mu\right) = 1 + r\sin^2\theta \,. \tag{65}$$

The pendulum swings down from $\theta{=}\pi/2$ to $\theta{=}\theta_-$) and back. One half of the pendulum's period is spent swinging down. Thus the full period is

$$T = \frac{2\ell}{v_0}\int_{\theta_-}^{\pi/2} d\theta \sqrt{\frac{1+r\sin^2\theta}{-\cot^2(\theta)+u\cos(\theta)}} \,. \tag{66}$$

$$= \frac{2\pi\ell}{v_0} f(u,r) \tag{67}$$

where $f(u,r)$ is a dimensionless function of u and r

Discussion: For simplicity let us set $r = m_2/m_1 = 1$. At small u, gravity's potential energy is very small compared the very large kinetic energy. Then the system makes very small oscillations between $\theta = \pi/2$ and $\theta = \pi/2 -$tinybit. The period of oscillations can be worked out analytically in this case leading to

$$T = \frac{2\pi\ell}{v_0}\sqrt{2} \qquad u \ll 1 \tag{68}$$

This is suggested as an exercise.

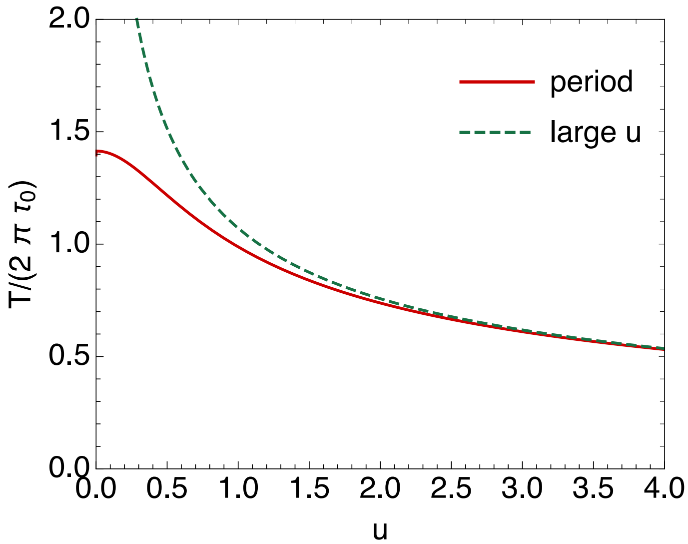


Figure 2: The period of the motion (normalized by $2\pi\ell/v_0$) as a function of u (see text).

At large u the system just falls, and the angular momentum can be neglected. Dimensional analysis in this limit says that the period is proportional to $\sqrt{\ell/g}$:

$$T = 2\pi\sqrt{\frac{\ell}{g}} \times \text{const} \tag{69}$$

This implies that $f(u,1) \to \text{const}/\sqrt{u}$ for $u \to \infty$. The integral can be done analytically in the limit that u is large, yielding

$$T = \frac{2\pi\ell}{v_0}\left(\frac{1.07}{\sqrt{u}}\right) \qquad u \gg 1 \tag{70}$$

at large u.

Fig. 2 shows the period as a function of u, and the limits we have outlined.