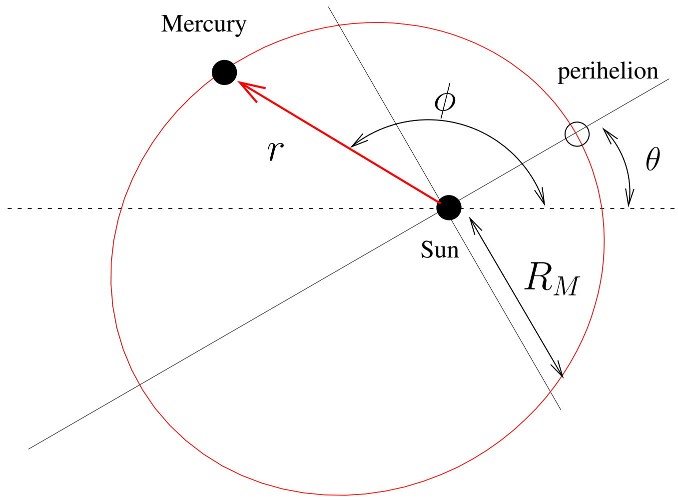
Problem 1. The precession of Mercury due to Jupiter

Recall that the trajectory of Mercury $r(\phi)$ is an ellipse with the sun at one focus as shown below. The perihelion (defined as the distance of closest approach) is rotated relative to the x-axis by an angle θ. The lattice rectum of Merucury is denoted R_M and is related to the angular momentum ℓ of the system (as discussed in class) but independent of the energy at fixed ℓ. The eccentricity of Mercury is small, $\epsilon = 0.2$, although it is the most eccentric of the Sun's planets.



Due to perturbations from the other planets, the angle of the perihelion θ changes (or precesses) as function of time. The precession rate is very small. The contribution of Jupiter to the precession rate is of order 150 arcsec/century, or (since the orbital period of Mercury is 88 days) approximately 1.78×10^{-6} rad/turn.

The goal of this problem is to estimate Jupiter's contribution to the precession rate[1]. Specifically, we will model Jupiter as a thin ring of mass M_J at the orbital radius of Jupiter R_J, and compute how this ring perturbs Mercury's orbit and causes the perihelion of Mercury to precess. Jupiter's orbital radius is significantly larger than Mercury's, $R_J \simeq 10\, R_M$.

(a) (Optional) Show that for $R_J \gg R_M$ the Lagrangian of Mercury interacting with the sun of mass $M_\odot$, and a ring of mass M_J and radius R_J is approximately

$$L \simeq \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2\,, \tag{1}$$

with $\alpha = GmM_J/(4R_J^3)$.

Hint: Let the origin be at the center of the ring. Let $\boldsymbol{r}$ be the vector from the center of the ring to a point of interest (i.e. Mercury) close to the center. For simplicity assume

[1] Famously, general relativity also perturbs the classical orbit and contributes 43 arcsecs/century to the total precession rate. This "anomalous" precession of Mercury was measured in the nineteenth century by le Verrier and finally explained by Einstein in 1915. The total precession rate is approximately 550 arcsec/century

that $\boldsymbol{r}$ lies in the xy plane. Then we want to integrate the Newton gravitational potential $d\Phi = -G dM_J/|\boldsymbol{r} - \boldsymbol{R}_J|$ over the mass of the ring to determine the gravitational potential due to the ring at point $\boldsymbol{r}$.

To this end, let $\boldsymbol{R}_J$ is a vector from the center to a point on the ring. Show that for $r \ll R_J$

$$\frac{1}{|\boldsymbol{r} - \boldsymbol{R}_J|} \simeq \frac{1}{R_J}\left(1 + \frac{r}{R_J}\cos(\phi) + \frac{r^2}{2R_J^2}\left(3\cos^2\phi - 1\right)\right) \tag{2}$$

where $\cos\phi$ is the angle between $\boldsymbol{r}$ and $\boldsymbol{R}_J$, and then integrate over ϕ.

(b) The orbit of mercury is characterized by its angular momentum ℓ. Since the unit of mass, time, and space are arbitrary we can (effectively) set three parameters to unity. Let us choose these three parameters to be ℓ, m, $k \equiv GM_\odot m$. Then all other scales are measured in these units.

(i) Construct a unit of length, R_0, time, T_0, and energy, E_0, with ℓ, m and k

(ii) Introduce a dimensionless radius $\underline{r} \equiv r/R_0$ and other suitable dimensionless variables, i.e. $\underline{t} \equiv t/T_0$ etc. Show that a dimensionless Lagrangian for the system is

$$\underline{L} = \frac{1}{2}\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\underline{\phi}}{d\underline{t}}\right)^2 + \frac{1}{\underline{r}} + \underline{\alpha}\,\underline{r}^2\,, \qquad \underline{L} = \frac{L}{E_0}\,, \tag{3}$$

where the dimensionless constant $\underline{\alpha}$ is of order

$$\underline{\alpha} \equiv \frac{M_J}{4M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 0.1 \times 10^{-6}\,. \tag{4}$$

The dimensional analysis step amounts to setting $\ell = m = GM_\odot m = 1$ everywhere in the original Lagrangian. We actually gained a little something by this analysis, i.e. without doing any computation we learned that the effect of the perturbing ring is of order one part in 10^7.

To lighten the notation, stop underlining the variables in what follows.

(iii) Show that the equations of motion for the dimensionless r and ϕ

$$\ddot{r} = -\frac{\partial V_{\text{eff}}}{\partial r}\,, \tag{5}$$

$$\dot{\phi} = \frac{1}{r^2}\,, \tag{6}$$

with the dimensionless effective potential is $V_{\text{eff}}(r) = 1/(2r^2) - 1/r - \alpha r^2$.

Recall from class that the eccentricity of the ellipse ($\alpha = 0$) is

$$e = \sqrt{1 + E/|E_{\min}|} \tag{7}$$

with $E_{\min} = \ell^2/(2mk)$. For the real "Mercurial" orbit the energy difference $\epsilon \equiv E - E_{\min} = e^2|E_{\min}| \simeq 0.04\,|E_{\min}|$ is small, and the orbit is nearly circular up to small oscillations of around the minimum of the effective potential. We will use an almost circular approximation for $\alpha \neq 0$, and evaluate the precession of the perihelion. Technically this involves expanding the effective potential $V_{\text{eff}}(r)$ around its (α dependent) minimum keeping terms linear in α.

(c) Determine the radius $r_{\min}(\alpha)$ for the circular orbit to first order in α. I find $r_{\min}(\alpha) \simeq 1 + 2\alpha$.

(d) Determine the period of radial oscillations for slight disturbances from this circular orbit of the previous part to first order in α. I find

$$\tau_M \simeq 2\pi(1 + 7\alpha) \tag{8}$$

(e) Show that the angle of perihelion of the ellipse θ will advance by an angle $\Delta\theta = 6\pi\alpha$ (see picture above), every time the particle reaches the distance of closest approach.

Restoring units we find

$$\Delta\theta = 6\pi\underline{\alpha} = \frac{3\pi}{2}\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 1.88 \times 10^{-6} \qquad \text{rad per turn}\,, \tag{9}$$

This should be compared to the experimental result of 1.78×10^{-6}rad/turn.

Problem 2. A scattering cross section

A particle of mass μ moves in the repulsive $1/r^2$ potential

$$U(r) = \frac{h}{r^2}\,, \quad h > 0\,.$$

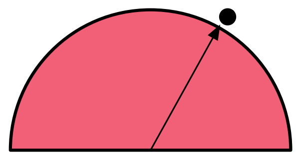
(a) Find equation for a generic trajectory $r(\phi)$ characterized with energy E and angular momentum $\ell \neq 0$. Follow the convention that the direction $\phi = 0$ points to the pericenter (point of closest approach).

(b) Find the time dependence on this trajectory, taking the time $t = 0$ at the pericenter.

(c) Find the differential scattering cross section $\frac{d\sigma(\theta)}{d\Omega}$ for a particle with energy E in this potential.

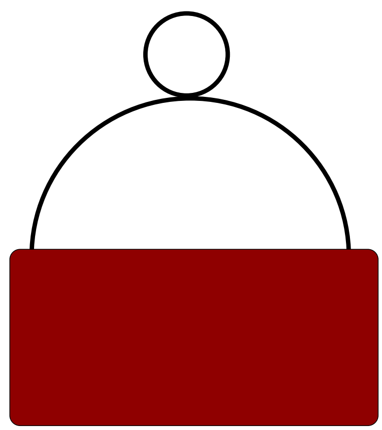
Problem 3. (Goldstein) A hoop on a cylinder

(a) First consider a small block of mass m on a cylinder of radius R on earth. If the block starts from rest on top of the cylinder, determine at what angle θ the block falls off the cylinder using the Lagrangian formalism to impose the constraint $r = R$.

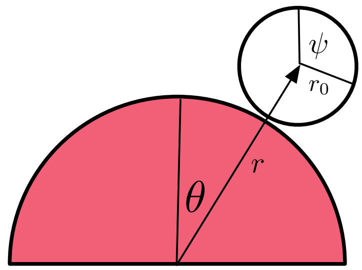


the coordinates are r, θ

(b) Now consider a hoop of mass m and radius r_0 rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder. You should find $\theta = 60^o$

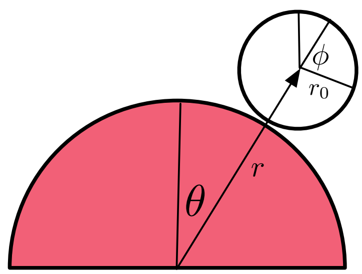


(i) Setup some coordinates. I took those based on the picture below. Determine the relaxation between the X and Y coordinates of a point on the rim of the hoop in terms of r, θ, ψ.



the coordinates are r, θ, ψ

An alternate choice of coordinates is to take an angle ϕ measured to the angle angle θ as shown below. You may wish to use the coordinates r, θ, ϕ, i.e. $\psi = \theta + \phi$



the coordinates are r, θ, ϕ

(ii) Starting with the general expression

$$T = \frac{1}{2} \int dm v^2 \tag{10}$$

show that the kinetic energy is of the hoop is

$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 \dot{\psi}^2 \tag{11}$$

In the alternate coordinates it reads

$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 (\dot{\theta} + \dot{\phi})^2 \tag{12}$$

(iii) Determine a relation between $d\theta$ and $d\psi$ if the hoop rolls without slipping. It may be easier to formulate a relation between $d\theta$ and $d\phi$.

(iv) Introducing a Lagrange multiplier for r to enforce the constraint (like in part a), and find the angle where the hoop falls off the cylinder. You should find $\theta_{\text{fall-off}} = 60^o$