Solution:

(a) The potential from a ring of mass of density $\rho(\boldsymbol{r}_o)$ is found from the Coulomb law like integral

$$\Phi(\boldsymbol{r}) = -\int_{\rm ring} d^3\boldsymbol{r}_o \, \frac{G\rho(\boldsymbol{r}_o)}{|\boldsymbol{r}-\boldsymbol{r}_o|} \, , \tag{1}$$

where $|\boldsymbol{r}_o^2| = R_J^2$ is the ring radius. Then expanding the denominator

$$\frac{1}{|\boldsymbol{r}-\boldsymbol{r}_o|} = \left(R_J^2 + r^2 - 2rR_J\cos\phi_o\right)^{-1/2} \, , \tag{2}$$

$$\simeq \frac{1}{R_J}\left[1 + \frac{r}{R_J}\cos\phi_o + \frac{r^2}{R_J^2}\left(\tfrac{3}{2}\cos^2\phi_o - \tfrac{1}{2}\right)\right] \, , \tag{3}$$

we obtain the integral expression for the potential

$$\Phi(r) = -\lambda G \int d\phi_o \left[1 + \frac{r}{R_J}\cos\phi_o + \left(\frac{r}{R_J}\right)^2\left(\tfrac{3}{2}\cos^2\phi_o - \tfrac{1}{2}\right)\right] \, , \tag{4}$$

where $\lambda = M_J/(2\pi R_J)$ is the linear mass density of the ring. After integrating over the angle ϕ_o, we find

$$\Phi = -\frac{GM_J}{R_J} - \frac{GM_J}{4R_J^3}r^2 \, . \tag{5}$$

Thus, the Lagrangian with the potential $V = m\Phi(\boldsymbol{r})$ is

$$L \simeq \frac{1}{2}m\dot{r}^2 + \frac{1}{2}mr^2\dot{\phi}^2 + \frac{GmM_\odot}{r} + \alpha r^2 \, , \tag{6}$$

where we have dropped an irrelevant constant, GmM_J/R_J, and

$$\alpha = \frac{GmM_J}{4R_J^3} \, . \tag{7}$$

(b) Then we rescale the radius

$$\underline{r} \equiv \frac{r}{R_0} \qquad R_0 = \frac{\ell^2}{mk} \, , \tag{8}$$

as given in the text. You can then find the timescale T_0 by directly determing which combo of ℓ, m, k has units of time. This is most easly done by noting that there is a timescale, T_0, where the kinetic term and potential terms of the unperturbed problem are the same order of magnitude

$$\frac{mR_0^2}{T_0^2} \sim \frac{k}{R_0} \, . \tag{9}$$

This leads us to define

$$T_0 \equiv \left(\frac{R_0^3 m}{k}\right)^{1/2} = \frac{\ell^3}{mk^2} \, . \tag{10}$$

The energy then is units of

$$E_0 = m\frac{R_0^2}{T_0^2} = \frac{k}{R_0} = \frac{k^2 m}{\ell^2} \tag{11}$$

Then the dimensionless Lagrangian is

$$\underline{L} = \frac{L}{E_o} = \frac{1}{2}\left(\frac{d\underline{r}}{d\underline{t}}\right)^2 + \frac{1}{2}\underline{r}^2\left(\frac{d\phi}{d\underline{t}}\right)^2 + \frac{1}{\underline{r}} + \underline{\alpha}\,\underline{r}^2\,, \tag{12}$$

where

$$E_o = \frac{k^2 m}{\ell^2} \tag{13}$$

and

$$\underline{\alpha} = \frac{\alpha R_0^2}{E_o} = \frac{1}{4}\left(\frac{M_J}{M_\odot}\right)\left(\frac{R_M}{R_J}\right)^3 . \tag{14}$$

(c) We drop the bars.

$$p_r = \dot{r} \qquad p_\phi = r^2\dot{\phi} \tag{15}$$

The Hamiltonian is

$$H = \frac{1}{2}p_r^2 + \frac{p_\phi^2}{2r^2} - \frac{1}{r} - \alpha r^2\,. \tag{16}$$

The Hamilton equations of motion are

$$\dot{r} = p_r\,, \tag{17}$$

$$\dot{\phi} = \frac{p_\phi}{r^2}\,, \tag{18}$$

$$\dot{p}_r = \frac{p_\phi^2}{r^3} - \frac{1}{r^2} + 2\alpha r\,, \tag{19}$$

$$\dot{p}_\phi = 0\,. \tag{20}$$

Now we set the initial value of $p_\phi(t_0) = \ell = 1$. The equations of motion guarantee that $p_\phi = 1$ at all subsequent times.

(d) We examine the radial equation,

$$\dot{p}_r = \frac{1}{r^3} - \frac{1}{r^2} + 2\alpha r\,, \tag{21}$$

and demand that the right hand side (the radial component of the force) is zero for circular orbits. Writing the circular radius as

$$r_o = r_o^{(0)} + r_o^{(1)}\,, \tag{22}$$

with $r_o^{(1)}$ small, we solve order by order in α. For instance we expand

$$\frac{1}{r^3} = \frac{1}{(r_o^{(0)} + r_o^{(1)})^3} \simeq \frac{1}{(r_o^{(0)})^3} - 3\frac{r_o^{(1)}}{(r_o^{(0)})^4} \tag{23}$$

Comparing the zero-th and 1st order equations we have

$$\text{0th order:} \qquad \frac{1}{(r_o^{(0)})^3} - \frac{1}{(r_o^{0})^2} = 0 \tag{24}$$

$$\text{1st order:} \qquad -3\frac{r_o^{(1)}}{(r_o^{(0)})^4} + \frac{2r_o^{(1)}}{(r_o^{(0)})^3} + 2\alpha r^{(0)} = 0 \tag{25}$$

The 0th order equation sets $r_o^{(0)} = 1$, and then the first order we find

$$r_o^{(1)} = 2\alpha \tag{26}$$

(e) We now expand the radius near the equilibrium circular radus, $r \to r_o + \delta r(t)$ with $r_o \equiv 1 + 2\alpha$. We find that to first order in δr and α we have

$$\frac{1}{(1+2\alpha+\delta r)^3} - \frac{1}{(1+2\alpha+\delta r)^2} + 2\alpha(1+\delta r) \simeq (1-14\alpha)\delta r \tag{27}$$

$$\frac{d^2\delta r}{dt^2} = \dot{p}_r = -(1-14\alpha)\delta r\,. \tag{28}$$

Thus, the period of radial oscillations is

$$\tau_M = \frac{2\pi}{\omega_0} = \frac{2\pi}{\sqrt{1-14\alpha}} \simeq 2\pi(1+7\alpha) \tag{29}$$

(f) Then we examine the angular equation

$$\dot{\phi} = \frac{1}{r^2}\,, \tag{30}$$

$$= \frac{1}{(r_o^{(0)} + r_o^{(1)} + \delta r)^2}\,, \tag{31}$$

$$\simeq (1 - 4\alpha - 2\delta r(t))\ . \tag{32}$$

Integrating the over a full period of radial oscillations $t = 0 \ldots \tau_M$, the term proportional to δr vanishes upon integration, and we find that the azimuthal angle has changed by

$$\Delta\phi = \tau_M(1 - 4\alpha\ell^6) \simeq 2\pi(1 + 3\alpha\ell^6)\,. \tag{33}$$

Thus the angle ϕ deviates from a full rotation by $+6\pi\alpha\ell^6$. The precession is

$$\Delta\phi = 6\pi\alpha \qquad \text{rad per turn}\,. \tag{34}$$

With the numerical value of α, we find

$$\Delta\phi = \frac{3\pi}{2}\frac{M_J}{M_\odot}\left(\frac{R_M}{R_J}\right)^3 \simeq 1.88\times10^{-6} \qquad \text{rad per turn}\,, \tag{35}$$

which should be compared to the nominal value of 1.78×10^{-6} rad/turn. Eq. (35) is 5% larger than the nominal value, and this deviation is consistent with order $\epsilon^2 \simeq 4\%$ corrections.

Problem 1. A scattering cross section

A particle of mass μ moves in the repulsive $1/r^2$ potential

$$U(r) = \frac{h}{r^2}\,, \quad h > 0\,.$$

(a) Find equation for a generic trajectory $r(\phi)$ characterized with energy E and angular momentum $\ell \neq 0$. Follow the convention that the direction $\phi = 0$ points to the pericenter (point of closest approach).

(b) Find the time dependence on this trajectory, taking the time $t = 0$ at the pericenter.

(c) Find the differential scattering cross section $\frac{d\sigma(\theta)}{d\Omega}$ for a particle with energy E in this potential.

Solution

For the repulsive potential, the energy E of the particle with non-vanishing angular momentum can only be positive. This means that there are no bound trajectories, or trajectories "falling on the center". There are only trajectories deflected by the center of the potential.

(a) To find such a trajectory quantitatively, we start with the integral for the polar angle ϕ and evaluate it:

$$\begin{aligned}\phi &= \frac{l}{(2\mu)^{1/2}} \int^r \frac{dr/r^2}{[E-(h+l^2/2\mu)/r^2]^{1/2}} \\ &= -\int^{1/r} \frac{du}{[(2\mu E/l^2)-(1+2\mu h/l^2)u^2]^{1/2}} = \\ &= \frac{1}{(1+2\mu h/l^2)^{1/2}} \arccos\Big(\frac{1/r}{\big[E/(h+l^2/2\mu)\big]^{1/2}}\Big).\end{aligned}$$

Rearranging the terms we finally get the explicit expression for the trajectory in the repulsive $1/r^2$ potential:

$$\frac{1}{r} = \Big(\frac{E}{h+l^2/2\mu}\Big)^{1/2} \cos\big[(1+2\mu h/l^2)^{1/2}\phi\big].$$

This expression shows that the fact that we have made an integration constant zero, when evaluating the integral for the trajectory, results in the standard choice of orientation of our polar system, in which the line to the pericenter is the direction $\phi = 0$. Also, we see that the distance to the pericenter is:

$$r_{min} = \Big(\frac{h+l^2/2\mu}{E}\Big)^{1/2}.$$

(b) For the $1/r^2$ potential, the integral for the time t along the trajectory, with the convention that $t=0$ at the pericenter, takes the form:

$$t = (\mu/2)^{1/2} \int_{r_{min}}^r \frac{dr}{[E-(h+l^2/2\mu)/r^2]^{1/2}},$$

and gives:

$$t = (\mu/8)^{1/2} \int_{r^2_{min}}^{r^2} \frac{dr^2}{[Er^2-(h+l^2/2\mu)]^{1/2}} = \Big(\frac{\mu}{2E}\Big)^{1/2}[r^2-r^2_{min}]^{1/2}.$$

This equation can be solved directly for r:

$$r(t) = \big[r^2_{min}+(v_0t)^2\big]^{1/2},$$

where $v_0 = (2E/\mu)^{1/2}$ is velocity of particle far away from the center. Combined with the equation for the trajectory obtained above, it also gives explicitly the time dependence of the polar angle:

$$\phi(t) = \frac{1}{(1+2\mu h/l^2)^{1/2}} \arcsin\Big(\frac{v_0t}{\big[r^2_{min}+(v_0t)^2\big]^{1/2}}\Big).$$

(c) The last equation in part (b) shows that the angle 2Ψ between the incoming and outgoing directions of the particle scattered by the $1/r^2$ potential is:

$$2\Psi = \frac{\pi}{(1 + 2\mu h/l^2)^{1/2}} \, .$$

This equation gives the relation between the scattering angle $\Theta = \pi - 2\Psi$ and the impact parameter b which determines the angular momentum, $l = \mu v_0 b$. An elementary algebra transforming this equation makes the relation explicit:

$$b = \frac{(h/E)^{1/2}(\pi - \Theta)}{\left[2\pi\Theta - \Theta^2\right]^{1/2}} \, .$$

From this relation we find the derivative of b:

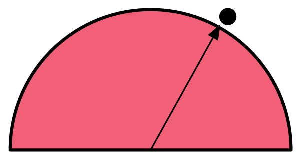
$$\left|\frac{db}{d\Theta}\right| = \frac{(h/E)^{1/2}\pi^2}{\left[2\pi\Theta - \Theta^2\right]^{3/2}} \, .$$

and finally, the differential cross section $d\sigma(\Theta)$:

$$\frac{d\sigma(\Theta)}{d\Omega} = \frac{b}{\sin\Theta}\left|\frac{db}{d\Theta}\right| = \frac{h}{E\sin\Theta}\,\frac{\pi^2(\pi - \Theta)}{\Theta^2(2\pi - \Theta)^2} \, .$$

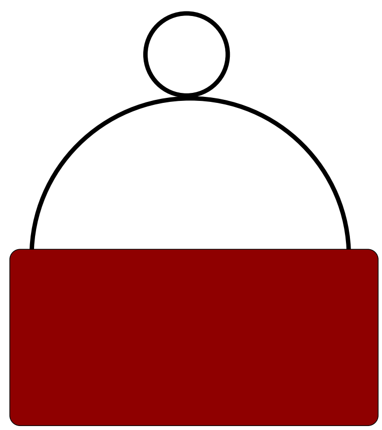
Problem 2. (Goldstein) A hoop on a cylinder

(a) First consider a small block of mass m on a cylinder of radius R on earth. If the block starts from rest on top of the cylinder, determine at what angle θ the block falls off the cylinder using the Lagrangian formalism to impose the constraint $r = R$.

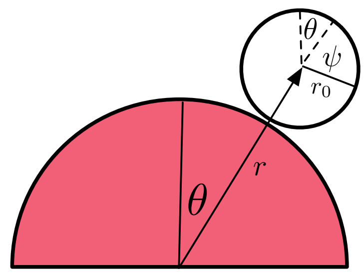


the coordinates are r, θ

(b) Now consider a hoop of mass m and radius r_0 rolls without slipping on a fixed cylinder of radius R as shown in the figure. The only external force is that of gravity. If the cylinder starts rolling from rest on top of the bigger cylinder, use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder. You should find $\theta = 60^o$

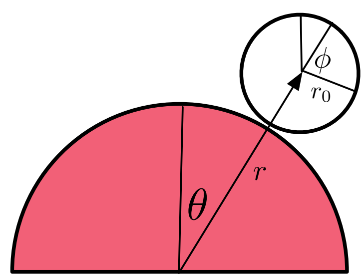


(i) Setup some coordinates. I took those based on the picture below. Determine the relaxation between the X and Y coordinates of a point on the rim of the hoop in terms of r, θ, ψ.



the coordinates are r, θ, ψ

An alternate choice of coordinates is to take an angle ϕ measured to the angle angle θ as shown below. You may wish to use the coordinates r, θ, ϕ, i.e. $\psi = \theta + \phi$



the coordinates are r, θ, ϕ

(ii) Starting with the general expression

$$T = \frac{1}{2} \int dm v^2 \tag{36}$$

show that the kinetic energy is of the hoop is

$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 \dot{\psi}^2 \tag{37}$$

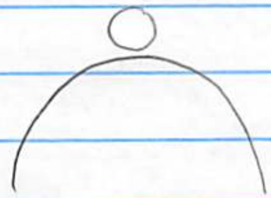
In the alternate coordinates it reads

$$T = \frac{1}{2} m r^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 (\dot{\theta} + \dot{\phi})^2 \tag{38}$$

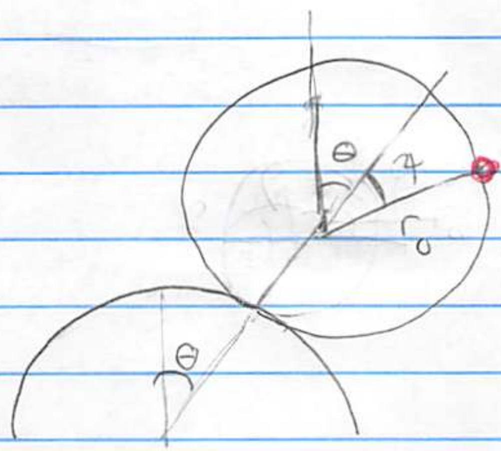
(iii) Determine a relation between $d\theta$ and $d\psi$ if the hoop rolls without slipping. It may be easier to formulate a relation between $d\theta$ and $d\phi$.

(iv) Introducing a Lagrange multiplier for r to enforce the constraint (like in part a), and find the angle where the hoop falls off the cylinder. You should find $\theta_{\rm fall-off} = 60^o$

A hoop on a cylinder



- We need to write down the Lagrangian, and thus need an appropriate set of coordinates;



- For a point on the hoop (see picture)

$$X = (R+r)\sin\theta + r_0 \sin\theta(\theta+\psi)$$

$$Y = +(R+r)\cos\theta + r_0\cos(\theta+\psi)$$

Here

$$\dot{X} = +(R+r)\cos\dot{\theta} + r_0\cos(\theta+\psi)(\dot{\theta}+\dot{\psi})$$

$$\dot{Y} = -(R+r)\sin\dot{\theta} - r_0\sin(\theta+\psi)(\dot{\theta}+\dot{\psi})$$

• So the kinetic energy evaluates to:

$$T = \int dm \, \frac{1}{2} \dot{x}^2 + \dot{y}^2$$

$$= \frac{m}{2\pi} \int_0^{2\pi} d\psi \quad \frac{1}{2} (\dot{x}^2 + \dot{y}^2)$$

$$= \underbrace{\frac{1}{2} m (R+r_0)^2 \dot{\theta}^2}_{\frac{1}{2} m v_{cm}^2} + \underbrace{\frac{1}{2} m r_0^2 (\dot{\theta}+\dot{\psi})^2}_{\frac{1}{2} I \omega^2}$$

where $I = m r_0^2$ and $\boxed{\omega = \dot{\theta} + \dot{\psi}}$

• Now there is a rolling without slipping constraint namely

$$R \, d\theta = r_0 \, d\psi \qquad \text{so} \qquad \omega = \dot{\theta} + \frac{R}{r_0} \dot{\theta} = \left(1 + \frac{R}{r_0}\right) \dot{\theta}$$

• Finally if r is not fixed but can change. KE is:

$$\boxed{T = \frac{1}{2} m (R+r)^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m r_0^2 \left(1 + \frac{R}{r_0}\right)^2 \dot{\theta}^2}$$

Now the CM is free to move but the angular motion is constrained $\theta = \frac{R}{r_0} \psi$. The

The

- internal kinetic energy is:

$$\left(\frac{1}{2} m r_0^2\right)\left(1+\frac{R}{r_0}\right)\dot{\theta}^2$$ describing a hoop

of radius r_0 whose rotational fequency is related to its CM motion $\omega = \left(1+\frac{R}{r_0}\right)\dot{\theta}$

- We will enforce that $r_0 = r$ with a lagrange multiplier. The constraint Lagrangian is

$$\hat{L} = \frac{1}{2} m (R+r)^2 \dot{\theta}^2 + \frac{1}{2} m \dot{r}^2 + \frac{1}{2} m (r_0+R)^2 \dot{\theta}^2$$
$$- m g (R+r) \cos\theta - \lambda (r - r_0)$$

So the EOM is

① $$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \theta}$$

$$\frac{d}{dt}\left[m(R+r)^2\dot{\theta} + m(r_0+R)^2\dot{\theta}\right] = + m g (R+r)\sin\theta$$

② $$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = \frac{\partial L}{\partial r}$$

$$m\ddot{r} = m(R+r)\dot{\theta}^2 - m g \cos\theta - \lambda$$

③ $\partial L / \partial \lambda = 0$ $\quad r = r_0$

- From ① and ③

$$\frac{d}{dt}\left[2m(R+r_0)^2\dot{\theta}\right] = -mg(R+r_0)\frac{\partial}{\partial\theta}\cos\theta$$

- This system has a "first integral" -- energy:

$$\frac{1}{2}2m(R+r_0)^2\dot{\theta}^2 + mg(R+r_0)\cos\theta = E$$

So

$$\dot{\theta}^2 = \left(E - mg(R+r_0)\cos\theta\right) / m(R+r_0)^2$$

- Since $\dot{\theta} = 0$ for $\theta = 0$, $E = mg(R+r_0)$ and

$$\dot{\theta}^2 = \cancel{m}g(R+r_0)(1-\cos\theta) / \cancel{m}(R+r_0)^2$$

From ② and ③ $\ddot{r} = 0$

$$0 = m(R+r)\dot{\theta}^2 - mg\cos\theta - \lambda$$

$$0 = mg(1-\cos\theta) - mg\cos\theta - \lambda$$

$$0 = mg - 2mg\cos\theta - \lambda$$

$$\lambda = mg - 2mg\cos\theta$$ ⟸ this is the normal force

The normal force vanishes for

$$\cos\theta = \frac{1}{2} \qquad \theta = 60^\circ$$ ← this is where it falls off!

- For the freshmann physics case -- a block on a cylinder -- the angle is

$$\theta = 2/3 \qquad \theta \simeq 48^\circ$$

- This make sense, all of the gravitational energy in the bead case gets converted into translational motion and the "centrifugal' force" is larger at smaller angles