Problem 1. Constraints in the Hamiltonian Formalism

If the canonical variables p_i, q_i, t (with $i = 1 \ldots N$) are not all independent but are related by auxiliary conditions of the form

$$\psi_k(p_i, q_i, t) = 0 \qquad (1)$$

(with $k = 1 \ldots m$) determine the modified Hamilton equations of motion by varying the appropriate action.

The solution to this problem is only a few lines.

Constraints in the Hamiltonian Formalism

$$\hat{S}[p,q,\lambda] = \int dt \; p_a \dot{q}^a - H(p,q) - \lambda \psi(p,q,t)$$

$$\delta\hat{S} = \int dt \Big[\delta p_a \dot{q}^a + p_a \frac{d\,\delta q^a}{dt} - \frac{\partial H}{\partial p_a}\delta p_a - \frac{\partial H}{\partial q^a}\delta q^a$$

$$- \delta\lambda\, \psi(p,q,t) - \lambda \frac{\partial \psi}{\partial p_a}\delta p_a - \lambda \frac{\partial \psi}{\partial q^a}\delta q^a \Big]$$

- Integrate me by parts and collect

$$\delta\hat{S} = \int dt \; \delta p_a \underline{\left(\dot{q}^a - \frac{\partial H}{\partial p_a} - \lambda \frac{\partial \psi}{\partial p^a} \right)} + \delta q^a \underline{\left(-\frac{dp_a}{dt} - \frac{\partial H}{\partial q^a} - \lambda \frac{\partial \psi}{\partial q^a} \right)}$$

$$+ \delta\lambda \; \underline{\psi(p,q)}$$

Setting the underlined terms to zero gives the EOM

Problem 2. The first order formalism and the transition to the Hamiltonian

This problem uses the notion of Lagrange multipliers and Legendre transforms to understand the action in the Hamiltonian formalism. Previously we said that the action in the Hamiltonian formalism is

$$S[q,p] = \int dt \left(p\frac{dq}{dt} - H(p,q) \right) . \tag{2}$$

We showed that extremizing this action gives Hamilton's equations of motion and that these equations are equivalent to the Euler-Lagrange equations. We did not, however, derive $S[q,p]$ directly from the action of the Lagrangian, $S[q]$. We will do this in this problem. The solution is only a few lines.

The action principle says that the action is

$$S[q] = \int dt L(q,\dot{q}) \tag{3}$$

and the system will follow the trajectory $\underline{q}(t)$ which extremizes this action. Using a Lagrange multiplier called $p(t)$ (for reasons discussed below), we may separately vary the velocity $v(t)$ and $\dot{q}$ by defining

$$\hat{S}[q(t),v(t),p(t)] \equiv \int dt\, \hat{L}(q,\dot{q},v,p) \qquad\qquad \hat{L} \equiv L(q,v) - p(v-\dot{q})\, . \tag{4}$$

and require that $\delta\hat{S} = 0$ for independent variations of q, v, p. The Lagrange multiplier enforces that $v = \dot{q}$ at the level of the equations of motion rather than the action. This "theorist-gone-wild" procedure is known as the "first order formalism", and has been found to be useful in analyzing various rich theories (such as gravity) which have complicated constraints.

(a) Consider the Lagrangian

$$L = \frac{1}{2}m\dot{q}^2 - U(q) \tag{5}$$

Show that the equations motion following from $\delta\hat{S}[q,v,p] = 0$, reproduce Newton's laws. Does the Lagrange multiplier have an appropriate name? Explain.

(b) One way to to extremize $\hat{S}$ is to first extremize $\hat{S}$ with respect to p, v with q fixed, leaving a reduced action $S_{\rm red}[q]$ to be extremized later. Argue that this reduced action is the Lagrangian formulation $S[q]$ in Eq. (3).

(c) Now extremize $\hat{S}$ with respect to v first with q and p fixed, leaving a reduced action $S_{\rm red}[q,p]$ to be extremized later, and argue that this reduced action is the Hamiltonian formulation $S[q,p]$ in Eq. (2).

First order formalism

$\boxed{a)}$ $\hat{L} = \frac{1}{2} m v^2 - U(q) - p(v - \dot{q})$

- So: $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q} \Rightarrow \frac{dp}{dt} = -\frac{\partial U}{\partial q}$

$$0 = \frac{\partial L}{\partial p} \qquad 0 = v - \dot{q}$$

$$0 = \frac{\partial L}{\partial v} \qquad 0 = mv - p$$

- These are the Lagrange EOM for $\hat{L}$, and they are equivalent to Newton Laws

$\boxed{b)}$ Now $\hat{S}[q, v, p] = \int dt \; L(q, v) - p(v - \dot{q})$

- Extremizing over p leaving v, q fixed:

$$\hat{S}[q, v, p + \delta p] - S[q, v, p] = -\int dt \; (v - \dot{q}) \; \delta p = 0$$

- So $\hat{S}$ is extremized at fixed v, q when

$$v = \dot{q}$$

and the thus extremized $\hat{S}$ is:

$$\hat{S}_{red}[q, \dot{q}, p] = \int dt \left(L(q, \dot{q}) - 0 \right) = \int dt\, L(q, \dot{q})$$

Leaving the Lagrangian form of the action

c) Similary the extremal point over v is

$$S[q, v + \delta v, p] = S[q, v, p] + \int dt \left(\frac{\partial L}{\partial v} - p \right) \delta v$$

- So now we solve for $v(p)$

$$\left. \frac{\partial L}{\partial v} \right|_{q\, v(p)} = p$$

- And substitute back in action:

$$\hat{S}_{red}[q, p] = \int dt\, \underbrace{\left[L(q, v(p)) - v(p)\, p \right]}_{\text{This is } -H} + p \frac{dq}{dt}$$

- So

$$\hat{S}_{red}[q, p] = \int dt\; p \frac{dq}{dt} - H(q, p)$$ ⟸ The usual Hamiltonian Action

Problem 3. (Milton, de Raad, Schwinger) Virial theorem from Noether logic

The virial theorem says that for the periodic motion of a particle the time averaged kinetic energy is related to an average of the potential energy:

$$\overline{2T} = \overline{\boldsymbol{r} \cdot \frac{\partial U(\boldsymbol{r})}{\partial \boldsymbol{r}}} \,. \tag{6}$$

For simplicity we will limit ourselves to the single particle Lagrangian

$$L = \frac{1}{2} m \dot{\boldsymbol{r}}^2 - U(\boldsymbol{r}) \,, \tag{7}$$

but when many particles are involved, the theorem generalizes straightforwardly .

$$\overline{2T} = \overline{\sum_a \boldsymbol{r}_a \cdot \frac{\partial U(\boldsymbol{r})}{\partial \boldsymbol{r}_a}} \,. \tag{8}$$

Here we will derive this useful result using Noether logic[1].

Recall that we say that the trajectory is called "onshell" if it satisfies the equation of motion, and, when necessary, notate this by placing a bar underneath the coordinates $\underline{\boldsymbol{r}}(t)$

(a) For a closed orbit of potential $U(r) \propto r^{\beta}$ what is the statement of the virial theorem. What is the statement of the theorem for a harmonic oscillator $U(r) \propto r^2$ and the gravitational potential $U(r) \propto r^{-1}$.

(b) Consider a quantum mechanical particle in one dimension in an energy eigenstate $H\,|\psi_n(x)\rangle = E_n\,|\psi_n\rangle$ (an eigenstate is analogous to the classical periodic trajectory). Show that for this eigenstate we have

$$\langle 2T \rangle = \left\langle x \frac{\partial U(x)}{\partial x} \right\rangle \tag{9}$$

by considering $\langle \psi_n|\,[xp, H]\,|\psi_n\rangle$. (Incidentally we will see later in the course that the generator $G(x,p) = xp$ generates infinitesimal rescalings in the classical theory. That it why it is natural, see below, to consider the commutator $[G, H]$ in the quantum mechanical formulation.)

(c) Now return to classical mechanics. Consider a specific variation of the trajectory consisting of an infinitesimal rescaling of the coordinate $\boldsymbol{r}$

$$\boldsymbol{r} \to (1+\epsilon)\boldsymbol{r} \tag{10}$$

What is the change of the onshell action $S[\underline{\boldsymbol{r}}]$ for this specific variation over one complete period of a periodic classical trajectory $\underline{r}$?

(d) What is the change in the action $\delta S[\boldsymbol{r}, \delta\boldsymbol{r}]$ for the specific variation in Eq. (10). Do not assume that $\boldsymbol{r}$ is onshell.

(e) Using (*c*) and (*d*) prove the theorem in Eq. (6)

[1] It is not exactly the Noether theorem, since there is no conserved charge and no symmetry. But the derivation is essentially the same as is used to derive Noether theorem.

Virial Theorem

a) If $U(r) = Cr^{\beta}$ then

$$\overline{2T} = \overline{r\frac{\partial V}{\partial r}} = \beta\ \overline{Cr^{\beta}}$$

Or $\overline{2T} = \beta\ \overline{U(r)}$

- So for a harmonic oscillator $\overline{T} = \overline{U}$, while for a r^{-1} potential $\overline{T} = -\frac{1}{2}\overline{U}$

b) Computing

$$\langle\psi|[px, H]|\psi\rangle = \langle px\rangle E_n - E_n\langle px\rangle = 0$$

But since $[ab, c] = a[b,c] + [a,c]b$

$$\left[px, \frac{p^2}{2m} + U(x)\right] = p[x, H] + [p, H]x$$

$$= i\hbar\frac{p^2}{m} - i\hbar\frac{\partial U}{\partial x}x$$

Leading to the theorem

$$\left\langle\frac{p^2}{m}\right\rangle = \left\langle x\frac{\partial U}{\partial x}\right\rangle$$

$\boxed{c)}$ Now consider the variation

$$\vec{r} \to \vec{r} + \delta_s \vec{r} \qquad \delta_s \vec{r} = \varepsilon \vec{r}$$

- So onshell:

$$\delta S[\underline{r}, \delta_s r] = \vec{p} \cdot \delta_s \vec{r} \Big|_{t_1}^{t_2} = \varepsilon \, \vec{p} \cdot \vec{r} \Big|_{t=t_2} - \varepsilon \vec{p} \cdot \vec{r} \Big|_{t=t_1}$$

- But since the motion is periodic, the state at time t_2 is the same as at t_1 and thus

$$\boxed{S[\underline{r}, \delta_s r] = 0}$$

$\boxed{d)}$ Now Lets consider the action

$$S[r + \varepsilon r] = \int dt \; L(r + \varepsilon r, \dot{r} + \varepsilon \dot{r})$$

So expanding

$$\delta S[r, \varepsilon r] = \int dt \; \frac{\partial L}{\partial \dot{r}} \varepsilon \dot{r} + \frac{\partial L}{\partial r} \varepsilon r$$

$$= \varepsilon \int dt \; m \dot{r}^2 - \frac{\partial U}{\partial r} r$$

e)

So restricting r onshell in d we find

$$\boxed{\overline{m\dot{r}^2} - \overline{r\,\partial U/\partial r} = 0}$$

Problem 4. Foucault Pendulum and the Coriolis Effect (MIT-OCW)

(a) We showed in class using the Newtonian formalism that, in a rotating frame of reference with ω constant, the equation of motion for a particle takes the form

$$m\boldsymbol{a}_r = \boldsymbol{F}_{\text{eff}} \,, \tag{11}$$

where

$$\boldsymbol{F}_{\text{eff}} \simeq \boldsymbol{F} - 2m\,(\boldsymbol{\omega} \times \boldsymbol{v}_r) \tag{12}$$

Here $\boldsymbol{v}_r = (dr^a/dt)\,\boldsymbol{e}_a(t)$ is the velocity in the rotating frame, and $\boldsymbol{e}_a(t)$ is the rotating basis of the frame. We have neglected terms of order ω^2 for simplicity. Derive this equation of motion from the Lagrangian formalism, where the Lagrangian in a fixed inertial frame is

$$L = \frac{1}{2}m\boldsymbol{v}_O^2 - U(\boldsymbol{r}) \tag{13}$$

with $\boldsymbol{v}_O = d\boldsymbol{r}/dt$ in that fixed frame.

Now consider a pendulum consisting of a long massless rod of length ℓ attached to a mass m. The pendulum is hung in a tower that is at latitude λ on the earth's surface[2], so it is natural to describe its motion with coordinates fixed to the rotating Earth. Let ω (i.e. once per day) be the Earth's angular velocity. Use either the (x, y, z) or (r, θ, ϕ) coordinates shown in the figure. Here z is perpendicular to the Earth's surface and y is tangent to a circle of constant longitude that passes through the north pole, and x therefore points east. The radius of the earth is R_e

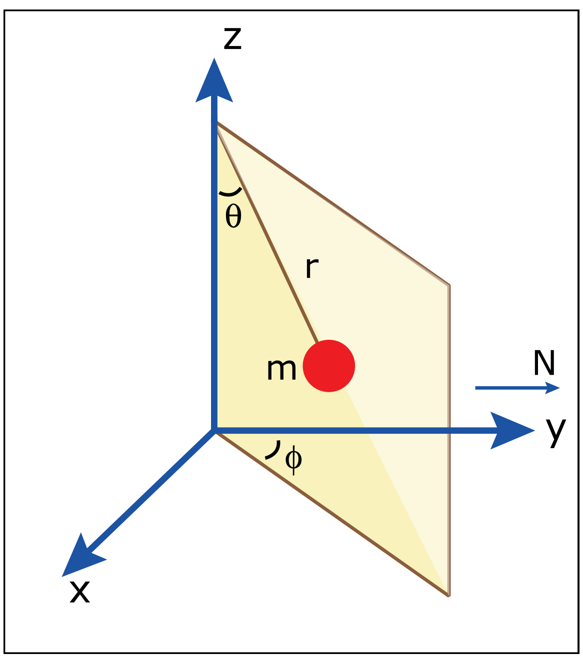

Image by MIT OpenCourseWare.

[2] 0^o latitude is the equator, 90^o latitude is the north pole

(b) Determine the Lagrangian of the Pendulum. From the start you may keep terms up to first order in ω, and of course you may neglect total time derivatives to simplify the analysis. Derive the Lagrangian for the pendulum small oscillations. I find

$$L = \frac{1}{2}m\ell^2\left[(\dot{\theta})^2 + \theta^2\dot{\phi}^2\right] - m\omega\ell^2\dot{\phi}\sin(\lambda)\theta^2 \;\; - mg\ell\frac{\theta^2}{2} \tag{14}$$

though in retrospect it may have been easier to use the xy coordinate system.

(c) Demonstrate that the pendulum undergoes precession with a rate $\dot{\phi} = \omega\sin\lambda$, by exactly solving the equations of motion for the small oscillations. Hint: it may be helpful to change variables back to Cartesian coordinates

$$x \equiv \ell\theta\sin(\phi) \tag{15}$$

$$y \equiv \ell\theta\cos(\phi) \tag{16}$$

before determining the equations of motion. The resulting equations can be solved exactly, by introducing $z(t) = x + iy$, and solving for z. Then the x and y coordinates may be recovered by taking the real and imaginary parts. Describe carefully which way the pendulum precesses.

(a) The velocity of a particle in a rotating frame is

$$\boldsymbol{v}_O = \boldsymbol{v} + \boldsymbol{\omega} \times r \tag{17}$$

Where here and below we will drop the label r, i.e. $\boldsymbol{v} = \boldsymbol{v}_r$.

So the Lagrangian is

$$L(\boldsymbol{r}, \dot{r}) = \frac{1}{2} m(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r})^2 - U(\boldsymbol{r}) \tag{18}$$

So the Lagrangian to first order in ω is

$$L = \frac{1}{2} m(\boldsymbol{v} + \boldsymbol{\omega} \times \boldsymbol{r})^2 - U(\boldsymbol{r}) \tag{19}$$

$$L \simeq \frac{1}{2} m\boldsymbol{v}^2 + m\boldsymbol{v} \cdot (\boldsymbol{\omega} \times \boldsymbol{r}) - U(\boldsymbol{r}) \tag{20}$$

$$L(r^a, \dot{r}^a) = \frac{1}{2} \dot{r}_a^2 + m\dot{r}^a \epsilon_{abc} \omega^b r^c - U(r^a) \tag{21}$$

In the last step we have expressed the Langrangian in terms of our chosen cooridinates (i.e. the coordinates r^a in the rest frame), rather than the vector notation. From here on it is just an ordinary Lagrangian mechanics prolbem, and the equations of motion are found in the usuual way

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}^d}\right) = \frac{\partial L}{\partial r^d} \tag{22}$$

$$\frac{d}{dt}(m\dot{r}_d + m\epsilon_{dbc}\omega^b r^c) = -\frac{\partial U}{\partial r^d} + m\dot{r}^a \epsilon_{abd} \omega^b \tag{23}$$

$$\frac{d}{dt}(m\dot{r}_d) = -\frac{\partial U}{\partial r^d} - 2m\epsilon_{dbc}\omega^b \dot{r}^c \tag{24}$$

This says that the d-th components satisfy

$$m(\boldsymbol{a})_d = -(\nabla_{\boldsymbol{r}} U)_d - 2m(\boldsymbol{\omega} \times \boldsymbol{v})_d \tag{25}$$

for all d, i.e. a vector equation is satisfied

$$m\boldsymbol{a} = -\nabla_{\boldsymbol{r}} U - 2m\,\boldsymbol{\omega} \times \boldsymbol{v} \tag{26}$$

(b) We parametrize the pendulum with the with the coordinates x, y

$$z = (\ell^2 - x^2 - y^2)^{1/2} = \ell - \frac{x^2 + y^2}{2\ell} \tag{27}$$

We then have

$$\frac{1}{2} m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \simeq \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) \tag{28}$$

Then at linear order

$$\boldsymbol{r} = (x, y, O(x^2)) \tag{29}$$

$$\boldsymbol{v} = (\dot{x}, \dot{y}, O(x^2)) \tag{30}$$

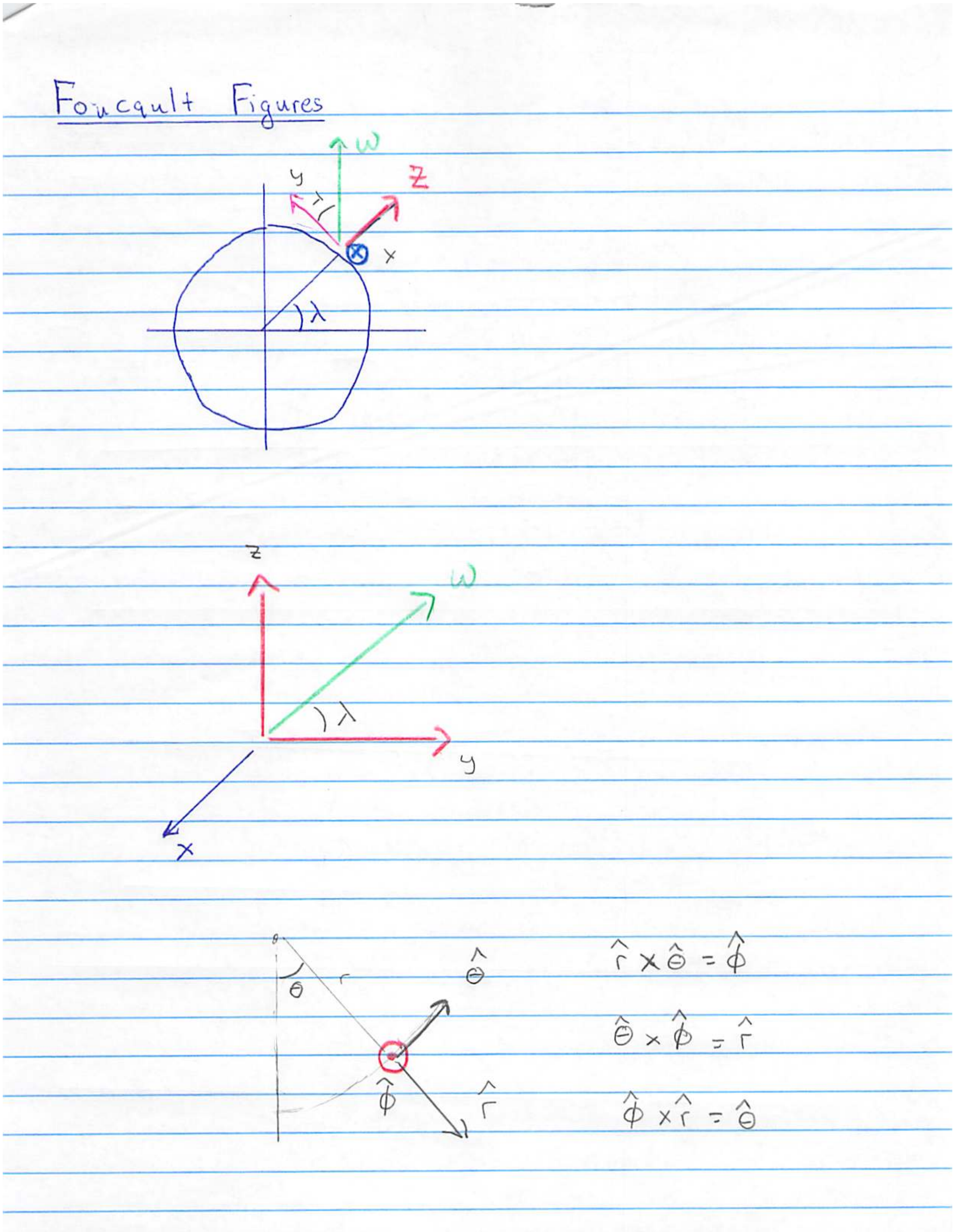

Figure 1: Figures for the focault pendulum.

Then using the picture on the next page we write

$$\boldsymbol{\omega} = (0, \omega\cos(\lambda), \omega\sin\lambda) \tag{31}$$

So

$$m\boldsymbol{\omega}\cdot(\boldsymbol{r}\times\boldsymbol{v}) = m\omega_0\sin\lambda(x\dot{y} - y\dot{x}) \tag{32}$$

So the Lagrangrian is simply

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - m\omega_0\sin(\lambda)(y\dot{x} - x\dot{y}) - \frac{1}{2}\frac{mg}{\ell}(x^2+y^2) \tag{33}$$

(c) The equations of motion stemming from the Lagrangian are

$$\ddot{x} = -\frac{g}{\ell}x + 2\omega_0\sin(\lambda)\dot{y} \tag{34}$$

$$\ddot{y} = -\frac{g}{\ell}y - 2\omega_0\sin(\lambda)\dot{x} \tag{35}$$

Note in particular the factor of 2 which arises from the Euler-Lagrange equations with $p = \partial L/\partial\dot{x} = m\dot{x} - m\omega_0\sin\lambda y$, e.g.

$$\frac{d}{dt}\left(m\dot{x} - m\omega_0\sin(\lambda)y\right) = -\frac{mg}{\ell}x + m\omega_0\sin(\lambda)\dot{y}\,, \tag{36}$$

$$\tag{37}$$

Combining into single equation for $\mathcal{Z} = y + ix$ we have

$$\ddot{\mathcal{Z}} = -\frac{g}{\ell}z + i2\omega_0\sin(\lambda)\dot{\mathcal{Z}} \tag{38}$$

For small ω_0 we can solve this euation (compare with the damped oscillator). By substituting $z = z_0e^{-i\omega t}$ we find for the characteristic frequencies

$$-\omega^2 + \frac{g}{\ell} - 2\omega_0\sin(\lambda)\omega = 0 \tag{39}$$

The roots of the characteristic equation are

$$\omega \simeq \pm\sqrt{\frac{g}{\ell}} - \omega_0\sin\lambda \tag{40}$$

So the solution is

$$(y+ix) = e^{+i\omega_0\sin\lambda t}\left(C_1e^{-i\sqrt{g/\ell}\,t} + C_2e^{i\sqrt{g/\ell}\,t}\right) \tag{41}$$

To understand the physics take the case when $C_1 = C_2 = A$, and then

$$\rho(t)e^{i\phi} = y + ix = e^{i\omega_0\sin\lambda t}A\cos(\sqrt{g/\ell}\,t)\,, \tag{42}$$

where $\tan\phi = x/y$, and $\rho = A\cos(\sqrt{g/\ell}t)$. The solution exhibits a precession in a westerly direction (increasing x) when the pendulum plane runs from north to south (the y direction). The precession rate is

$$\dot{\phi} = \omega_0\sin\lambda \tag{43}$$