## Problem 1. Preliminaries

Answer as briefly as possible! Just a few lines each – enough to show you know how it works and no more.

(a) (Optional, but strongly recommended if not obvious to you) Give an informal explanation (given in class) why

$$\epsilon_{abc}\epsilon_{\underline{a}\underline{b}c} = \delta_{a\underline{a}}\delta_{b\underline{b}} - \delta_{a\underline{b}}\delta_{b\underline{a}} \tag{1}$$

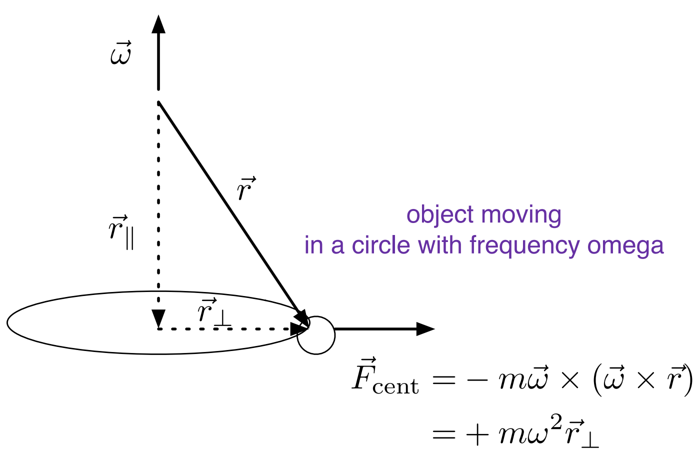
and use it to prove the "bac" to "abc" rule[1]:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a}\cdot\boldsymbol{c}) - (\boldsymbol{a}\cdot\boldsymbol{b})\boldsymbol{c} \tag{2}$$

The centrifugal force is

$$\boldsymbol{F}_{\rm cent} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,. \tag{3}$$

Use the *bac abc* rule to show that $\boldsymbol{F}_{\rm cent} = m\omega^2\boldsymbol{r}_\perp$ where $\boldsymbol{r}_\perp$ is the part of $\boldsymbol{r}$ perpendicular to $\boldsymbol{\omega}$. The figure below shows how this is used.




(b) Given a tensor $\boldsymbol{I} = I_{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ in the rotating basis and in the fixed basis[2] $\boldsymbol{I} = \underline{I}_{ab}\underline{\boldsymbol{e}}_a \otimes \underline{\boldsymbol{e}}_b$ (here $\boldsymbol{e}_a = R_{ab}\underline{\boldsymbol{e}}_b$), show that the components are related via

$$I_{ab} = R_{ac}\,R_{bd}\,\underline{I}_{cd}\,. \tag{4}$$

Express this transformation rule with matrices.

(c) Show that

$$\boldsymbol{w} \times \boldsymbol{v} = \hat{\boldsymbol{v}} \cdot \boldsymbol{w} = \boldsymbol{v} \cdot \hat{\boldsymbol{w}} \tag{5}$$

where (for example) $\hat{\boldsymbol{v}} = \hat{v}_{ab}\;\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ denotes the antisymmetric tensor $\hat{v}_{ab} = \epsilon_{abc}v^c$ associated with the vector $\boldsymbol{v}$. Express these two alternate forms of the cross product using matrices.

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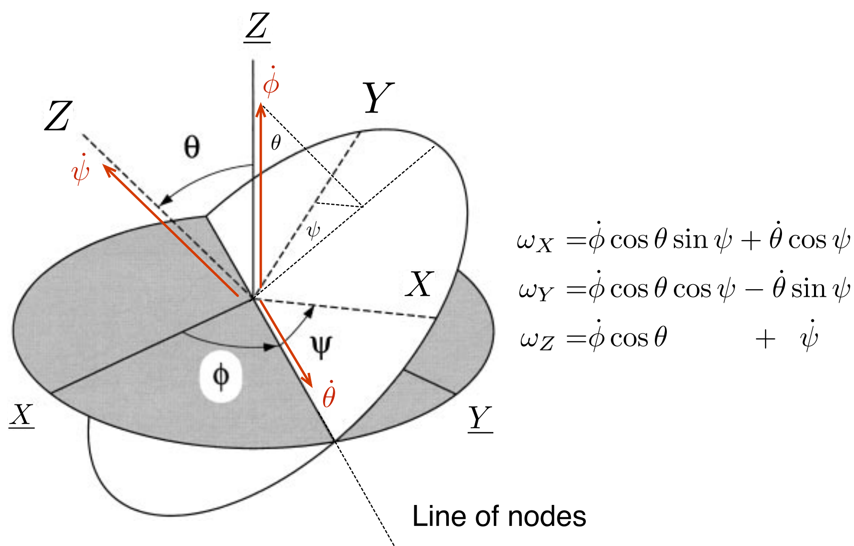
[1] read as "back to abc".

[2] Often I will write $\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ as simply $\boldsymbol{e}_a\boldsymbol{e}_b$ with the $\otimes$ implied. Then $\boldsymbol{I}\cdot\boldsymbol{v}$ takes the dot product with the second slot $\boldsymbol{I}\cdot\boldsymbol{v} = I_{ab}v^b\,\boldsymbol{e}_a$, while $\boldsymbol{v}\cdot\boldsymbol{I}$ takes the dot product with the first, $v^aI_{ab}\boldsymbol{e}_b$.

(d) Show that $\underline{\omega}_{ac} = (R^{-1}\dot{R})_{ac}$

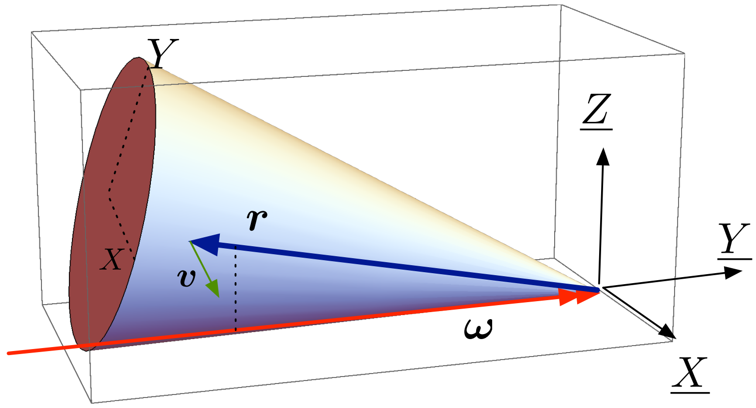
(e) Determine the projection of $\vec{\omega}$ on to the lab frame axes $\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3$. (You may use either algebraic means, computer algebraic means, or use the appropriate picture from lecture, or all three means.) You should find

$$\begin{pmatrix} \omega_{\underline{X}} \\ \omega_{\underline{Y}} \\ \omega_{\underline{Z}} \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\phi) \\ \dot{\theta}\sin(\phi) - \dot{\psi}\sin(\theta)\cos(\phi) \\ \dot{\psi}\cos(\theta) + \dot{\phi} \end{pmatrix} \tag{6}$$



## Problem 2. A Rolling Cone (Adapted from Geldstein Ch.5 #17)

A uniform right circular cone of height $h$, half-angle $\alpha$, and density $\rho$ rolls on its side without slipping on a uniform horizontal plane. It returns to its original position in a time $\tau$ .




(a) Find the moment of inertia tensor for the body (or principal) axes centered on the tip. I find

$$I^0 = \frac{3}{5}Mh^2 \begin{pmatrix} \frac{1}{4}\tan^2\alpha + 1 & & \\ & \frac{1}{4}\tan^2\alpha + 1 & \\ & & \frac{1}{2}\tan^2\alpha \end{pmatrix} \tag{7}$$

(b) The cone is turning around the $\underline{Z}$ axis in a counterclockwise fashion as seen from above. Consider the infinitesimal rotation at $t = 0$ (see figure) that the cone experiences – the displacement of a point $\boldsymbol{r}$ on the cone's body is

$$\boldsymbol{r} \to \boldsymbol{r} + \delta\boldsymbol{\theta} \times \boldsymbol{r}\,, \tag{8}$$

where $\delta\boldsymbol{\theta}$ points along the $\underline{Y}$ axis. Describe qualitatively why Eq. (8) (with the specified direction of $\boldsymbol{\omega}$) is what we mean by a rolling cone. Argue in particular that $\underline{\omega}_z = 0$ and write down the components of $\boldsymbol{\omega}(t)$ in the lab frame.

(c) Determine the Euler angles describing the cone as a function of time. Take the $Z$ axis to point along the axle of the cone. Interpret $\dot{\phi}$ and the relation between $\dot{\psi}$ and $\dot{\phi}$.

(d) Find the kinetic energy of the rolling cone. I find

$$T = Mh^2 \left(\frac{2\pi}{\tau}\right)^2 \left[\frac{3}{40}(1 + 5\cos^2\alpha)\right] \tag{9}$$

(e) (Optional.) Write down the components of the $\boldsymbol{L}(t)$ in the lab frame. (You may wish to check your results by computing $T = \frac{1}{2}\boldsymbol{\omega} \cdot \boldsymbol{L}$)

(f) (Optional.) There are two ways to compute the kinetic energy. The first way uses the expression

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot I_{\text{tip}} \cdot \boldsymbol{\omega}\,. \tag{10}$$

where $I_{\text{tip}}$ is the moment of inertia around the tip. The second way uses the moment of inertia of the center of masss $I_{\text{cm}}$

$$T = \frac{1}{2}\boldsymbol{\omega} \cdot I_{\text{cm}} \cdot \boldsymbol{\omega} + \frac{1}{2}M\boldsymbol{v}_{\text{cm}}^2 \,. \tag{11}$$

Show that these are equivalent to each other provided $I_{\text{cm}}$ and $I_{\text{tip}}$ are related by the parallel axis theorem.

## Problem 3. Nutation of a Heavy Symmetric Top

Consider a heavy symmetric top with one end point fixed.

(a) Write down the Lagrangian from class. Carry out Routh's procedure explicity by Legendre transforming with respect to the the conserved momenta $p_\psi$ and $p_\phi$. Write down $-R$ which serves as effective Lagrangian $L_{\rm eff}$ for $\theta$. Show that $\theta$ obeys the equation of motion following from this effective Lagrangian

$$I\ddot{\theta} = -\frac{\partial U_{\rm eff}}{\partial \theta}\,, \tag{12}$$

where

$$U_{\rm eff} = mg\ell\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta}\,. \tag{13}$$

Also show that

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2(\theta)}\,. \tag{14}$$

(b) In class we analyzed the limit when gravitational torque is small to the rotational kinetic energy, $mg\ell/(p_\psi^2/I_1) \ll 1$. Take $p_\phi/p_\psi = r$ with $0 < r < 1$. Within this approximation (known as the fast top approximation), if the energy $E$ is adjusted to the minimum of the effective potential, the tip of the top will slowly precess with

$$\dot{\theta} = 0\,, \quad \text{and} \quad \dot{\phi} = \frac{mg\ell}{p_\psi}\,. \tag{15}$$

This is is shown in Fig. 1(d) which shows the trajectory of the tip of the top on the sphere.

Now if the energy of the system is slightly larger than the minimum of $U_{\rm eff}$, describe qualitatively the motion in $\theta$ and $\phi$. For what range in $E$ do the first (a) and second (b) figures describe the top's motion? Explain. Work in the fast top approximation

(c) Using the fast top approximation outlined in ($b$), compute the period of $\theta$ oscillations for a given energy $E$ with $E$ just larger than the minimum of $U_{\rm eff}$. Determine the precession rate $\dot{\phi}(t)$, as a function of time. You should find

$$\dot{\phi} = \frac{mg\ell}{p_\psi} - \frac{p_\psi}{I_1}\frac{A}{\sin^2\theta_0}\cos(\omega_0 t) \tag{16}$$

where $\omega_0 = p_\psi/I_1$, and $\theta_0$ is the mean value of the small $\delta\theta$.

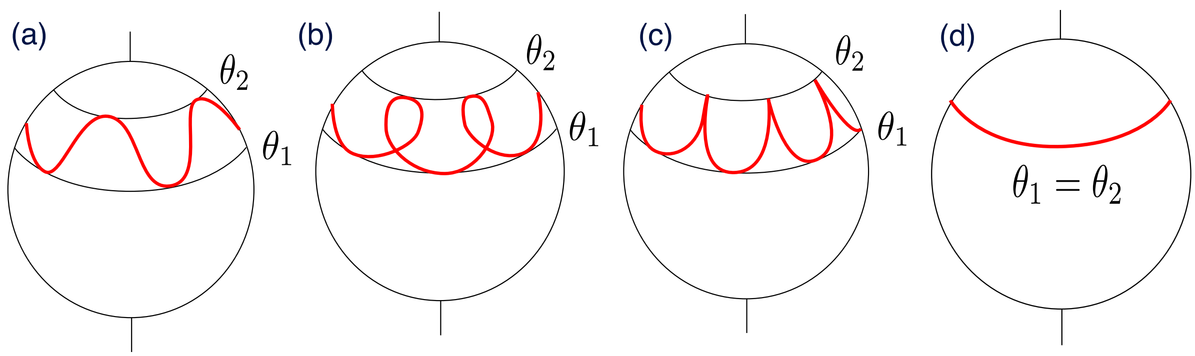



Figure 1: Motion of the tip of the heavy symmetric top