Problem 1. Preliminaries

Answer as briefly as possible! Just a few lines each – enough to show you know how it works and no more.

(a) (Optional, but strongly recommended if not obvious to you) Give an informal explanation (given in class) why

$$\epsilon_{abc}\epsilon_{\underline{a}\underline{b}c} = \delta_{a\underline{a}}\delta_{b\underline{b}} - \delta_{a\underline{b}}\delta_{b\underline{a}} \tag{1}$$

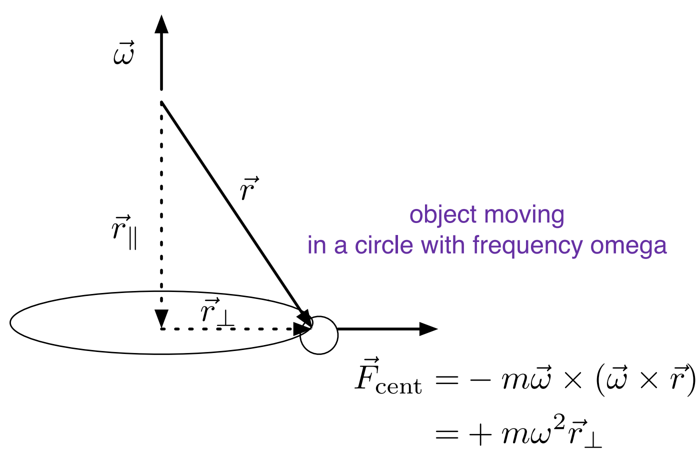
and use it to prove the "bac" to "abc" rule[1]:

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \tag{2}$$

The centrifugal force is

$$\boldsymbol{F}_{\rm cent} = -m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \boldsymbol{r})\,. \tag{3}$$

Use the *bac abc* rule to show that $\boldsymbol{F}_{\rm cent} = m\omega^2\boldsymbol{r}_\perp$ where $\boldsymbol{r}_\perp$ is the part of $\boldsymbol{r}$ perpendicular to $\boldsymbol{\omega}$. The figure below shows how this is used.



(b) Given a tensor $\boldsymbol{I} = I_{ab}\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ in the rotating basis and in the fixed basis[2] $\boldsymbol{I} = \underline{I}_{ab}\underline{\boldsymbol{e}}_a \otimes \underline{\boldsymbol{e}}_b$ (here $\boldsymbol{e}_a = R_{ab}\underline{\boldsymbol{e}}_b$), show that the components are related via

$$I_{ab} = R_{ac}\,R_{bd}\,\underline{I}_{cd}\,. \tag{4}$$

Express this transformation rule with matrices.

(c) Show that

$$\boldsymbol{w} \times \boldsymbol{v} = \hat{\boldsymbol{v}} \cdot \boldsymbol{w} = \boldsymbol{v} \cdot \hat{\boldsymbol{w}} \tag{5}$$

where (for example) $\hat{\boldsymbol{v}} = \hat{v}_{ab}\;\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ denotes the antisymmetric tensor $\hat{v}_{ab} = \epsilon_{abc}v^c$ associated with the vector $\boldsymbol{v}$. Express these two alternate forms of the cross product using matrices.

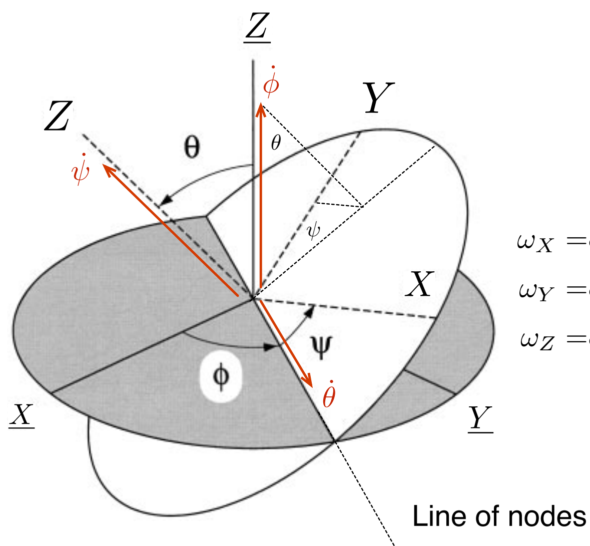
[1] read as "back to abc".

[2] Often I will write $\boldsymbol{e}_a \otimes \boldsymbol{e}_b$ as simply $\boldsymbol{e}_a\boldsymbol{e}_b$ with the $\otimes$ implied. Then $\boldsymbol{I} \cdot \boldsymbol{v}$ takes the dot product with the second slot $\boldsymbol{I} \cdot \boldsymbol{v} = I_{ab}v^b\,\boldsymbol{e}_a$, while $\boldsymbol{v} \cdot \boldsymbol{I}$ takes the dot product with the first, $v^a I_{ab}\boldsymbol{e}_b$.

(d) Show that $\underline{\omega}_{ac} = (R^{-1}\dot{R})_{ac}$

(e) Determine the projection of $\vec{\omega}$ on to the lab frame axes $\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3$. (You may use either algebraic means, computer algebraic means, or use the appropriate picture from lecture, or all three means.) You should find

$$\begin{pmatrix} \omega_{\underline{X}} \\ \omega_{\underline{Y}} \\ \omega_{\underline{Z}} \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\phi) \\ \dot{\theta}\sin(\phi) - \dot{\psi}\sin(\theta)\cos(\phi) \\ \dot{\psi}\cos(\theta) + \dot{\phi} \end{pmatrix} \tag{6}$$



$$\omega_X = \dot{\phi}\cos\theta\sin\psi + \dot{\theta}\cos\psi$$
$$\omega_Y = \dot{\phi}\cos\theta\cos\psi - \dot{\theta}\sin\psi$$
$$\omega_Z = \dot{\phi}\cos\theta \quad + \quad \dot{\psi}$$

(a) Just us $\boldsymbol{e}_a = R_{ac}\underline{\boldsymbol{e}}_c$, and find immediately

$$\boldsymbol{I} = I_{ab}R_{ac}R_{bd}\underline{\boldsymbol{e}}_c \otimes \underline{\boldsymbol{e}}_d = \underline{I}_{cd}\underline{\boldsymbol{e}}_c \otimes \underline{\boldsymbol{e}}_d \tag{7}$$

So

$$I_{ab}R_{ac}R_{bd} = \underline{I}_{cd} \tag{8}$$

Or in matrix form

$$(R^T I R)_{cd} = (\underline{I})_{cd} \tag{9}$$

Multiplying on the left and right by matrices R and R^T respectively gives

$$(I)_{ab} = (R\underline{I}R^T)_{ab} \tag{10}$$

Or reverting to index notation

$$I_{ab} = R_{ac}\underline{I}_{cd}(R^T)_{db} \tag{11}$$

$$= R_{ac}R_{bd}\underline{I}_{cd} \tag{12}$$

(b) We have

$$\boldsymbol{w} \times \boldsymbol{v} = \epsilon_{abc} w_b v_c \boldsymbol{e}_a \tag{13}$$

$$= \hat{v}_{ab} w_b \boldsymbol{e}_a \tag{14}$$

$$= \hat{\boldsymbol{v}} \cdot \boldsymbol{w} \tag{15}$$

$$= v_c \epsilon_{cab} w_b \boldsymbol{e}_a \tag{16}$$

$$= v_c \hat{w}_{ca} \boldsymbol{e}_a \tag{17}$$

$$= \boldsymbol{v} \cdot \hat{\boldsymbol{w}} \tag{18}$$

(c) We have from part a

$$(\underline{\hat{\omega}})_{cd} = (R^T \hat{\omega} R)_{cd} \tag{19}$$

Since $\hat{\omega} = \dot{R}R^{-1}$ we have

$$(\underline{\hat{\omega}})_{cd} = (R^T \dot{R} R^{-1} R)_{cd} = (R^{-1}\dot{R})_{cd} \tag{20}$$

(d) First we use algebra. The matrix R_{ab} reads

$$\begin{pmatrix} \cos(\phi)\cos(\psi) - \cos(\theta)\sin(\phi)\sin(\psi) & \cos(\psi)\sin(\phi) + \cos(\theta)\cos(\phi)\sin(\psi) & \sin(\theta)\sin(\psi) \\ -\cos(\theta)\cos(\psi)\sin(\phi) - \cos(\phi)\sin(\psi) & \cos(\theta)\cos(\phi)\cos(\psi) - \sin(\phi)\sin(\psi) & \cos(\psi)\sin(\theta) \\ \sin(\theta)\sin(\phi) & -\cos(\phi)\sin(\theta) & \cos(\theta) \end{pmatrix} \tag{21}$$

Then we simply compute $\dot{R}_{ab}R^{-1}_{bc} = \dot{R}_{ab}(R^T)_{bc}$ yielding

$$\hat{\omega} = \begin{pmatrix} 0 & \cos(\theta)\phi' + \psi' & \sin(\psi)\theta' - \cos(\psi)\sin(\theta)\phi' \\ -\cos(\theta)\phi' - \psi' & 0 & \cos(\psi)\theta' + \sin(\theta)\sin(\psi)\phi' \\ \cos(\psi)\sin(\theta)\phi' - \sin(\psi)\theta' & -\cos(\psi)\theta' - \sin(\theta)\sin(\psi)\phi' & 0 \end{pmatrix} \tag{22}$$

We can read from this matrix the corresponding entries of $\vec{\omega} = (\hat{\omega}_{23}, \hat{\omega}_{31}, \hat{\omega}_{12})$

$$\omega_x = \cos(\psi)\theta' + \sin(\theta)\sin(\psi)\phi' \tag{23}$$

$$\omega_y = -\sin(\psi)\theta' + \cos(\psi)\sin(\theta)\phi' \tag{24}$$

$$\omega_z = \cos(\theta)\phi' + \psi' \tag{25}$$

Now we can project ω on the $\underline{\boldsymbol{e}}$ axes

$$\vec{\omega} = \omega_a \boldsymbol{e}_a = \omega_a R_{ab} \underline{\boldsymbol{e}}_b = \omega_b \underline{\boldsymbol{e}}_b \tag{26}$$

So without thinking we type the appropriate expression into mathematica

$$\underline{\omega}_b = \omega_a R_{ab} \tag{27}$$

As a matrix

$$\begin{pmatrix} \underline{\omega}_x & \underline{\omega}_y & \underline{\omega}_z \end{pmatrix} = \begin{pmatrix} \omega_x & \omega_y & \omega_z \end{pmatrix} \begin{pmatrix} & & \\ & R & \\ & & \end{pmatrix} \tag{28}$$

yielding

$$\begin{pmatrix} \underline{\omega}_x & \underline{\omega}_y & \underline{\omega}_z \end{pmatrix} = \begin{pmatrix} \theta' \cos(\phi) + \psi' \sin(\theta)\sin(\phi) \\ \theta' \sin(\phi) - \psi' \sin(\theta)\cos(\phi) \\ \psi' \cos(\theta) + \phi' \end{pmatrix}^T \tag{29}$$

This is also clear from the picture given in class (with one extra line added). For instance from Fig. 1, we have

$$\underline{\omega}_Z = \dot{\psi}\cos\theta + \dot{\phi} \tag{30}$$

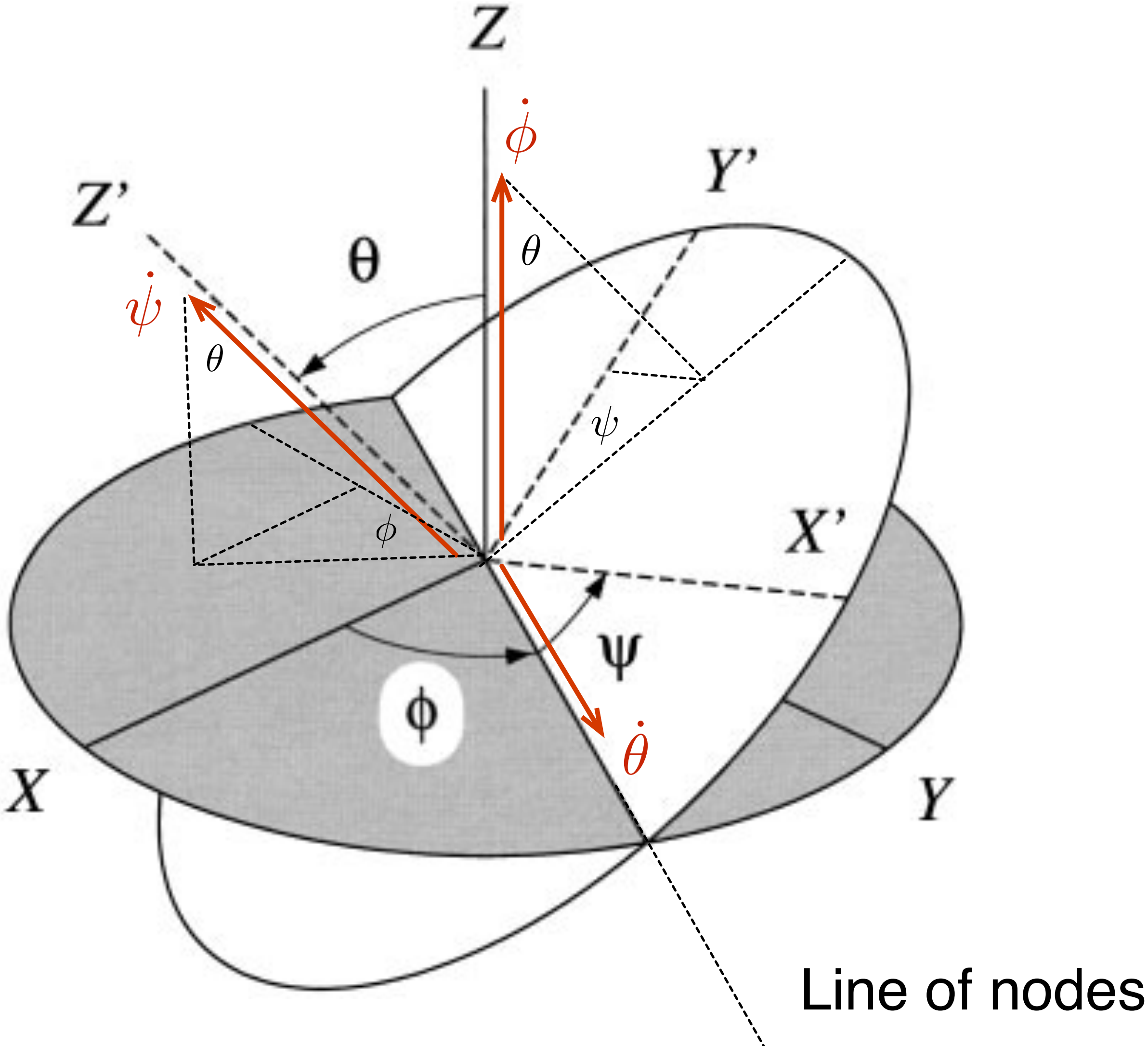
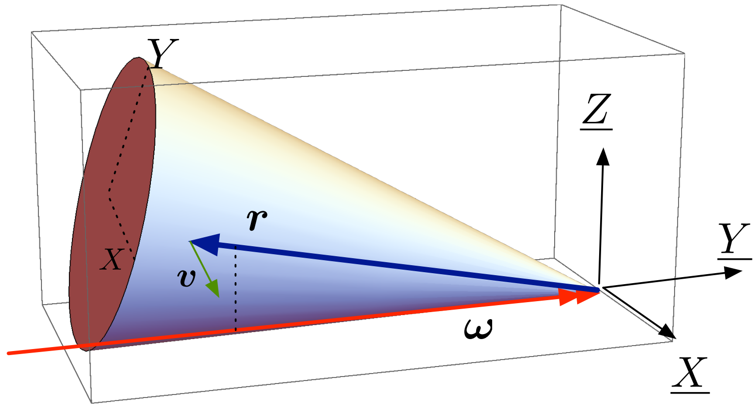


Figure 1:

Problem 2. A Rolling Cone (Adapted from Geldstein Ch.5 #17)

A uniform right circular cone of height h, half-angle α, and density ρ rolls on its side without slipping on a uniform horizontal plane. It returns to its original position in a time τ .



(a) Find the moment of inertia tensor for the body (or principal) axes centered on the tip. I find

$$I^0 = \frac{3}{5}Mh^2 \begin{pmatrix} \frac{1}{4}\tan^2\alpha + 1 & & \\ & \frac{1}{4}\tan^2\alpha + 1 & \\ & & \frac{1}{2}\tan^2\alpha \end{pmatrix} \tag{31}$$

(b) The cone is turning around the $\underline{Z}$ axis in a counterclockwise fashion as seen from above. Consider the infinitesimal rotation at $t = 0$ (see figure) that the cone experiences – the displacement of a point $\boldsymbol{r}$ on the cone's body is

$$\boldsymbol{r} \to \boldsymbol{r} + \delta\boldsymbol{\theta} \times \boldsymbol{r}\,, \tag{32}$$

where $\delta\boldsymbol{\theta}$ points along the $\underline{Y}$ axis. Describe qualitatively why Eq. (32) (with the specified direction of $\boldsymbol{\omega}$) is what we mean by a rolling cone. Argue in particular that $\underline{\omega}_z = 0$ and write down the components of $\boldsymbol{\omega}(t)$ in the lab frame.

(c) Determine the Euler angles describing the cone as a function of time. Take the Z axis to point along the axle of the cone. Interpret $\dot{\phi}$ and the relation between $\dot{\psi}$ and $\dot{\phi}$.

(d) Find the kinetic energy of the rolling cone. I find

$$T = Mh^2 \left(\frac{2\pi}{\tau}\right)^2 \left[\frac{3}{40}(1 + 5\cos^2\alpha)\right] \tag{33}$$

(e) (Optional.) Write down the components of the $\boldsymbol{L}(t)$ in the lab frame. (You may wish to check your results by computing $T = \frac{1}{2}\boldsymbol{\omega}\cdot\boldsymbol{L}$)

(f) (Optional.) There are two ways to compute the kinetic energy. The first way uses the expression

$$T = \frac{1}{2}\boldsymbol{\omega}\cdot I_{\rm tip}\cdot\boldsymbol{\omega}\,. \tag{34}$$

where $I_{\rm tip}$ is the moment of inertia around the tip. The second way uses the moment of inertia of the center of masss $I_{\rm cm}$

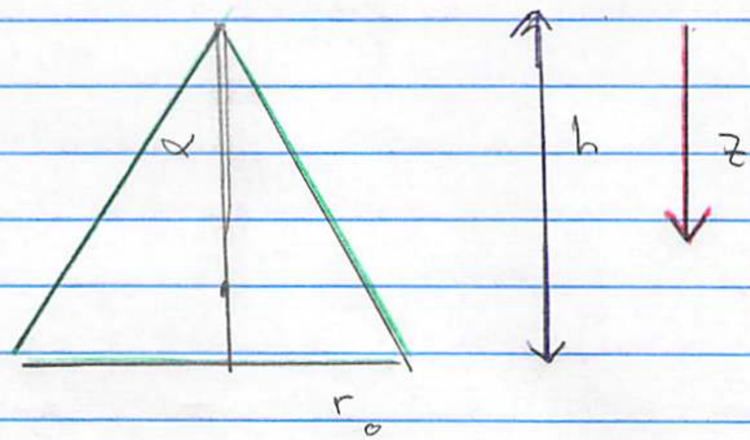
$$T = \frac{1}{2}\boldsymbol{\omega} \cdot I_{\rm cm} \cdot \boldsymbol{\omega} + \frac{1}{2}M\boldsymbol{v}_{\rm cm}^2 \,. \tag{35}$$

Show that these are equivalent to each other provided $I_{\rm cm}$ and $I_{\rm tip}$ are related by the parallel axis theorem.

(a) First I find the center of mass, the moment of inertia around the tip, and the moment of inertia around the center of mass. The center of mass questions were not asked for, but was asked for in other years so I include it here as example

A Rolling Cone

a) The center of mass:



- $r(z) = z \tan\alpha$

$$\text{Volume} = \frac{\pi}{3} h^3 \tan^2\alpha$$

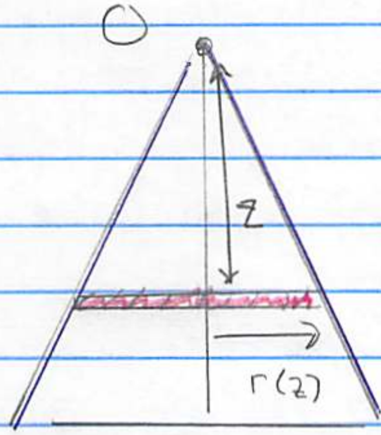
$$z_{cm} = \int_0^h dm\, z \;/\; \rho V$$

Then $dm = \pi r^2(z)\, dz\, \rho$

$$z_{cm} = \frac{\int_0^h \rho \pi z^3 \tan^2\alpha \, dz}{\int_0^h \rho \pi z^2 \tan^2\alpha \, dz} = \frac{1/4\, h^4}{1/3\, h^3}$$

$$\boxed{z_{cm} = 3/4\, h}$$

- To find the moment of inertia around the tip, one finds the moment of inertia tensors from the disk and then add them up.



- The moments of inertia of the disk around its center of mass is:

$$I^{cm}_{disk} = \begin{pmatrix} 1/4\, dm\, r^2 & & \\ & 1/4\, dm\, r^2 & \\ & & +1/2\, dm\, r^2 \end{pmatrix}$$

- Then using the parallel axis theorem (see class!) the moment of inertia tensor about O is:

$$I^{O}_{disk} = \begin{pmatrix} 1/4\, dm\, r^2 + dm\, z^2 & & \\ & \frac{1}{4} dm\, r^2 + dm\, z^2 & \\ & & +\frac{1}{2} m\, dr^2 \end{pmatrix}$$

- Then Integrating

$$I^{O}_{cone} = \int I^{O}_{disk} \quad \text{with} \int dm = \int_0^h \rho\, \pi r^2(z)\, dz$$

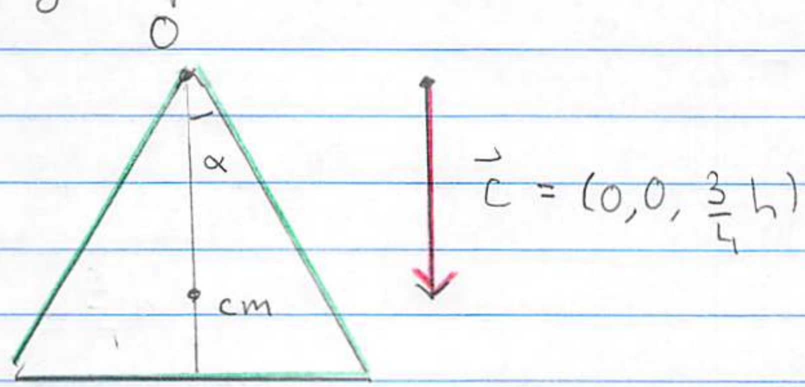
The integration is straight-forward and yields

$$I^O = \frac{3}{5} m h^2 \begin{pmatrix} \frac{1}{4}\tan^2\alpha + 1 & & \\ & \frac{1}{4}\tan^2\alpha + 1 & \\ & & 1/2 \tan^2\alpha \end{pmatrix}$$

(b) Around the CM we may use the Parallel axis theorem

$$(I^O)_{ab} = (I^{cm})_{ab} + m\,(c^2 \delta_{ab} - c_a c_b)$$

where $\vec{c} = (0, 0, \frac{3}{4}h)$ is the shift vector connecting point O with the CM

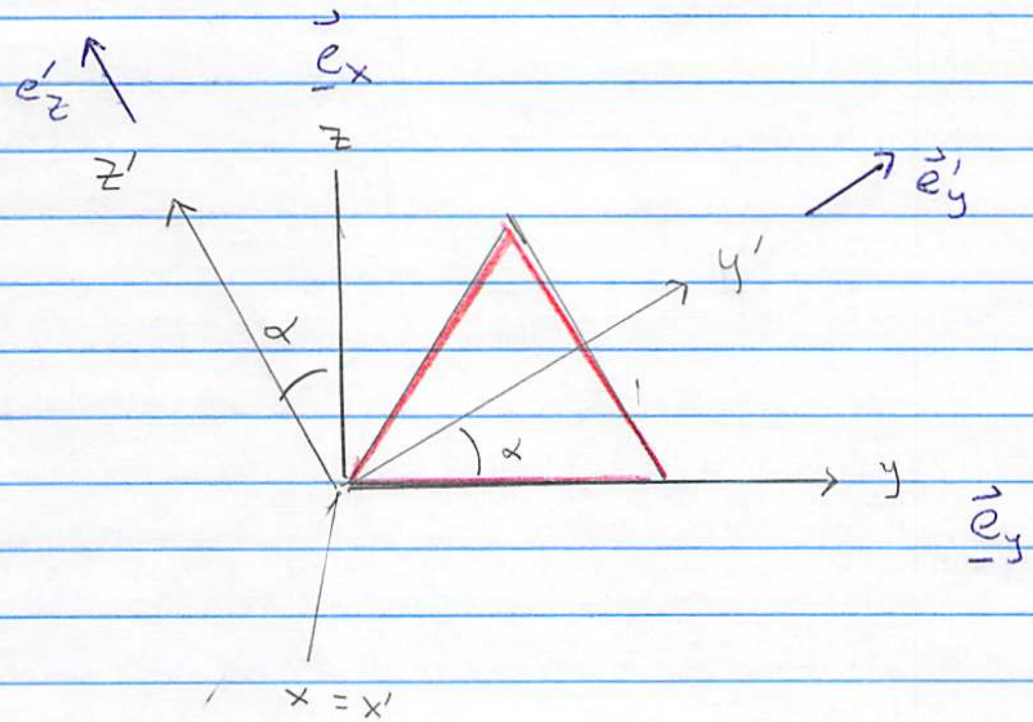


This gives

$$(I^{cm})_{ab} = I^0 - M \begin{pmatrix} (3/4)^2 h^2 & & \\ & (3/4\, h)^2 & \\ & & 0 \end{pmatrix}$$

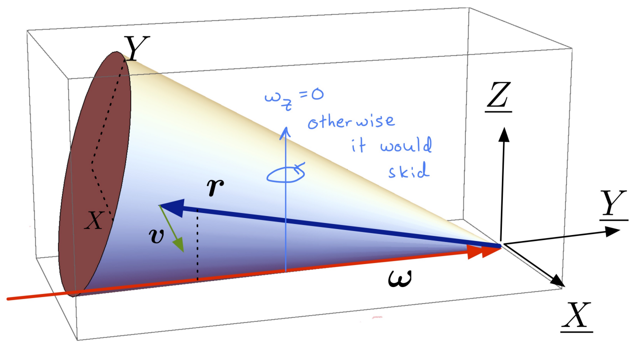
$$(I^{cm})_{ab} = \frac{3}{10} M h^2 \begin{pmatrix} \frac{1}{8} + \frac{1}{2}\tan^2\alpha & & \\ & \frac{1}{8} + \frac{1}{2}\tan^2\alpha & \\ & & \tan^2\alpha \end{pmatrix}$$

(c)



(b) In each time moment the cone pivots around the line of contact. Any rotation around the $\underline{Z}$ axis would cause a skidding motion of the line of contact (see figure), which is not allowed by the rolling without slipping constraint. Indeed the points on the line of contact are not moving. If there was an $\omega_{\underline{Z}}$ then these points would move in time δt by an amount $\delta \boldsymbol{r} = \omega_{\underline{Z}} \delta t \; \hat{\underline{\mathbf{z}}} \times \boldsymbol{r}$. We must in general have $\boldsymbol{\omega}$ parallel to the line of contact if the line of contact is to be stationary. The line of contact rotates with by an angle of 2π over time τ and this means

$$(\underline{\omega}_X, \underline{\omega}_Y, \underline{\omega}_Z) = \omega_0(-\sin(2\pi/\tau), \cos(2\pi/\tau), 0) \tag{36}$$

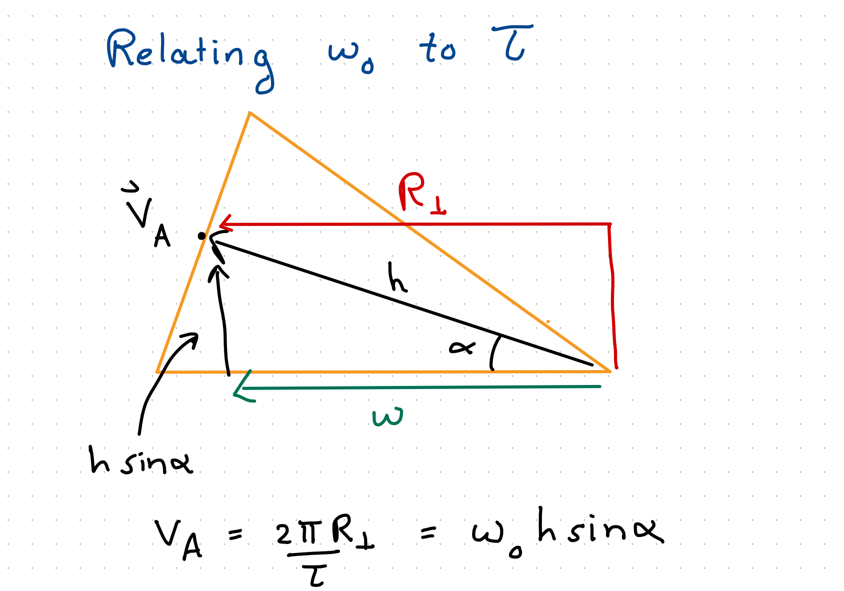


The last step is to relate ω_0 to τ. The point A moves in a circle of radius $R_\perp = h\cos\alpha$ over time τ and thus has velocity $v_A = 2\pi h \cos\alpha/\tau$. Since $v_A = |\boldsymbol{\omega} \times \boldsymbol{r}_A| = \omega_0 h \sin\alpha$ (see figure and remember we are rotating around the line of contact), we have $\omega_0 = \frac{2\pi}{\tau}\cot\alpha$, leading to

$$\underline{\omega}_X = \frac{2\pi}{\tau}\cot\alpha\,(-\sin(2\pi t/\tau)) \tag{37}$$

$$\underline{\omega}_Y = \frac{2\pi}{\tau}\cot\alpha\,(\cos(2\pi t/\tau)) \tag{38}$$

$$\underline{\omega}_Z = 0 \tag{39}$$



(c) Having found the the angular velocity we can find the Euler angles. We have from the previous excercise

$$\begin{pmatrix} \omega_{\underline{X}} \\ \omega_{\underline{Y}} \\ \omega_{\underline{Z}} \end{pmatrix} = \begin{pmatrix} \dot{\theta}\cos(\phi) + \dot{\psi}\sin(\theta)\sin(\phi) \\ \dot{\theta}\sin(\phi) - \dot{\psi}\sin(\theta)\cos(\phi) \\ \dot{\psi}\cos(\theta) + \dot{\phi} \end{pmatrix} \tag{40}$$

where for example $\theta' = d\theta/dt$, which should be matched with Eq. (37). So we need to solve this equations for θ, ϕ, and ψ. Using geometry we see first that $\theta = \pi/2 - \alpha$ is constant and thus

$$\sin(\theta) = \cos(\alpha)\,, \tag{41}$$

$$\cos(\theta) = \sin(\alpha)\,. \tag{42}$$

Thus we need

$$\dot{\phi} = \frac{2\pi}{\tau} \tag{43}$$

$$\dot{\psi} = -\frac{2\pi}{\tau}\frac{1}{\sin\alpha} \tag{44}$$

Or

$$\phi = \frac{2\pi t}{\tau} \tag{45}$$

$$\psi = -\frac{2\pi t}{\tau}\frac{1}{\sin\alpha} \tag{46}$$

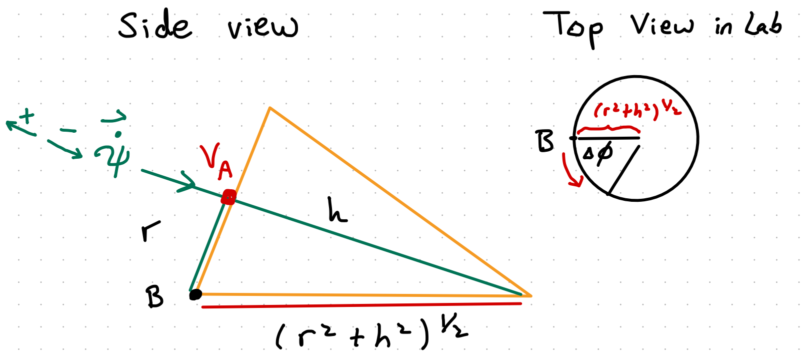
Clearly we interpret $\phi = 2\pi t/\tau$ as the azimuthal angle of the cone with respect to the fixed axes. Relation between $\dot{\phi}$ and $\dot{\psi}$ came from the rolling without slipping constraint

$$0 = d\psi\sin(\alpha) + d\phi \tag{47}$$

The sign is correct, since of the cone advances in by $d\phi$ then the vector

$$d\boldsymbol{\psi} \equiv dt\,\dot{\boldsymbol{\psi}} \equiv dt\,\dot{\psi}\,\boldsymbol{e}_z \tag{48}$$

to point towards the apex of the cone. The figure below interprets this constraint relation further



If we roll, then we have two expressions for the distance travelled by B, $r|d\psi| = (r^2+h^2)^{1/2}\, d\phi$, or $|d\psi| = d\phi/\sin\alpha$. Finally if point A comes out of the page then $\Delta\phi$ is positive but $\dot\psi$ points towards the tip of the cone, which is negative with our conventions.

(d) We first write down the components of $\boldsymbol{\omega}$ in the body basis. From the figure given in the problem

$$\omega_X = 0 \tag{49}$$

$$\omega_Y = \omega_0 \sin\alpha \tag{50}$$

$$\omega_Z = -\omega_0 \cos\alpha \tag{51}$$

with $\omega_0 = (2\pi/\tau)\cot\alpha$. Evaluating

$$T = \frac{1}{2} I_{YY}\omega_Y^2 + \frac{1}{2} I_{ZZ}\omega_Z^2 \tag{52}$$

where from part (a)

$$I_{yy} = \frac{3}{5} M h^2 (\tfrac{1}{4}\tan^2\alpha + 1) \qquad I_{zz} = \frac{3}{5} M h^2 (\tfrac{1}{2}\tan^2\alpha) \tag{53}$$

we find after minor algebra the result quoted in the problem

$$T = \frac{3}{40} M h^2 \left(\frac{2\pi}{\tau}\right)^2 (1 + 5\cos^2\alpha) \tag{54}$$

(e) To evaluate $\boldsymbol{L}$ in the lab frame we can simply evaluate $\boldsymbol{L}$ at $t = 0$ and then recognize that $\boldsymbol{L}$ at a later time is simply a rotated version of this

$$\boldsymbol{L}|_{t=0} = \boldsymbol{e}_z I_{zz}\omega_z + \boldsymbol{e}_y I_{yy}\omega_y \tag{55}$$

We can then use geometry

$$\boldsymbol{e}_z = \cos\theta\, \underline{\boldsymbol{e}}_z - \sin\theta\, \underline{\boldsymbol{e}}_y = \sin\alpha\, \underline{\boldsymbol{e}}_z - \cos\alpha\, \underline{\boldsymbol{e}}_y \tag{56}$$

$$\boldsymbol{e}_y = \sin\theta\, \underline{\boldsymbol{e}}_z + \cos\theta\, \underline{\boldsymbol{e}}_y = \cos\alpha\, \underline{\boldsymbol{e}}_z + \sin\alpha\, \underline{\boldsymbol{e}}_y \tag{57}$$

which after minor algebra gives

$$\boldsymbol{L}|_{t=0} = \frac{3}{40} M h^2 \dot{\phi} \tan\alpha \left[(2 + 10\cos^2\alpha) \underline{\boldsymbol{e}}_y + (10\cos^2\alpha - 2) e_z \right] \tag{58}$$

We can easily verify that with

$$\boldsymbol{\omega} = (0, \omega_0, 0) = (0, \dot{\phi}\cot\alpha, 0) \tag{59}$$

one recovers part (d)

$$T = \frac{1}{2} \boldsymbol{\omega} \cdot \boldsymbol{L} = \frac{3}{40} M h^2 \dot{\phi}^2 (1 + 5\cos^2\alpha) \tag{60}$$

To evaluate $\boldsymbol{L}(t)$ a later time $t \neq 0$ one simply makes the replacement,

$$\underline{\boldsymbol{e}}_y \to \cos(2\pi t/\tau) \underline{\boldsymbol{e}}_y - \sin(2\pi t/\tau) \underline{\boldsymbol{e}}_x \tag{61}$$

in Eq. (55).

Problem 3. Nutation of a Heavy Symmetric Top

Consider a heavy symmetric top with one end point fixed.

(a) Write down the Lagrangian from class. Carry out Routh's procedure explicity by Legendre transforming with respect to the the conserved momenta p_ψ and p_ϕ. Write down $-R$ which serves as effective Lagrangian $L_{\rm eff}$ for θ. Show that θ obeys the equation of motion following from this effective Lagrangian

$$I\ddot{\theta} = -\frac{\partial U_{\rm eff}}{\partial \theta}\,, \tag{62}$$

where

$$U_{\rm eff} = mg\ell\cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1\sin^2\theta}\,. \tag{63}$$

Also show that

$$\dot{\phi} = \frac{p_\phi - p_\psi\cos\theta}{I_1\sin^2(\theta)}\,. \tag{64}$$

(b) In class we analyzed the limit when gravitational torque is small to the rotational kinetic energy, $mg\ell/(p_\psi^2/I_1) \ll 1$. Take $p_\phi/p_\psi = r$ with $0 < r < 1$. Within this approximation (known as the fast top approximation), if the energy E is adjusted to the minimum of the effective potential, the tip of the top will slowly precess with

$$\dot{\theta} = 0\,, \quad \text{and} \quad \dot{\phi} = \frac{mg\ell}{p_\psi}\,. \tag{65}$$

This is is shown in Fig. 2(d) which shows the trajectory of the tip of the top on the sphere.

Now if the energy of the system is slightly larger than the minimum of $U_{\rm eff}$, describe qualitatively the motion in θ and ϕ. For what range in E do the first (a) and second (b) figures describe the top's motion? Explain. Work in the fast top approximation

(c) Using the fast top approximation outlined in (b), compute the period of θ oscillations for a given energy E with E just larger than the minimum of $U_{\rm eff}$. Determine the precession rate $\dot{\phi}(t)$, as a function of time. You should find

$$\dot{\phi} = \frac{mg\ell}{p_\psi} - \frac{p_\psi}{I_1}\frac{A}{\sin^2\theta_0}\cos(\omega_0 t) \tag{66}$$

where $\omega_0 = p_\psi/I_1$, and θ_0 is the mean value of the small $\delta\theta$.

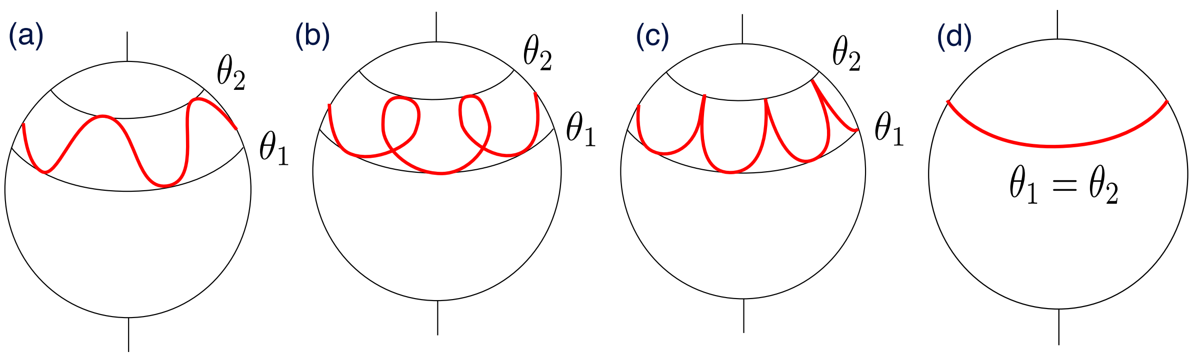


Figure 2: Motion of the tip of the heavy symmetric top

Solution:

(a) The first part of this is directly out of lecture and we will not analyze it further here, see page 6 of lecture.

(b) Now let us analyze the motion in the effective potential. The precession rate is determined by $\dot{\phi}$

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{I_1 \sin^2(\theta)} \,. \tag{67}$$

And the effective potential is

$$U_{\rm eff} = mg\ell \cos\theta + \frac{(p_\phi - p_\psi\cos\theta)^2}{2I_1 \sin^2\theta} \,. \tag{68}$$

We set up some dimensionless variables as done in lecture, effectively setting

$$I_1 = p_\psi = m = 1. \tag{69}$$

Motivating the definitions

$$\bar{g} \equiv \frac{mg\ell}{p_\psi^2/I_1} \tag{70}$$

$$r \equiv \frac{p_\phi}{p_\psi} \tag{71}$$

$$\bar{E} \equiv \frac{E}{p_\psi^2/I_1} \tag{72}$$

So the effective potential and presssion rate read

$$\dot{\phi} = \frac{(r - \cos\theta)}{\sin^2\theta} \tag{73}$$

while the effective potential is

$$U_{\rm eff} = \bar{g}\cos(\theta) + \frac{(r-\cos\theta)^2}{\sin^2\theta} \,. \tag{74}$$

Below we will work with $u = \cos\theta$ and the effective potential for the variable u is

$$\frac{U_{\rm eff}(u)}{p_\psi^2/I_1} \equiv \bar{U} = \bar{g}u + \frac{(r-u)^2}{1-u^2} \tag{75}$$

The precession rate is

$$\frac{\dot{\phi}}{p_\psi/I_1} = \frac{r-u}{1-u^2} \tag{76}$$

In the fast top approximation $\bar{g} \ll 1$. Then in the zeroth approximation we can neglect the $\bar{g}$ term. Then a sketch of the potential for $r > 0$ versus $u \equiv \cos\theta$. is shown on the next page. For $u > r$ the precession rate is positive while for $u < r$ the precession rate is negative.

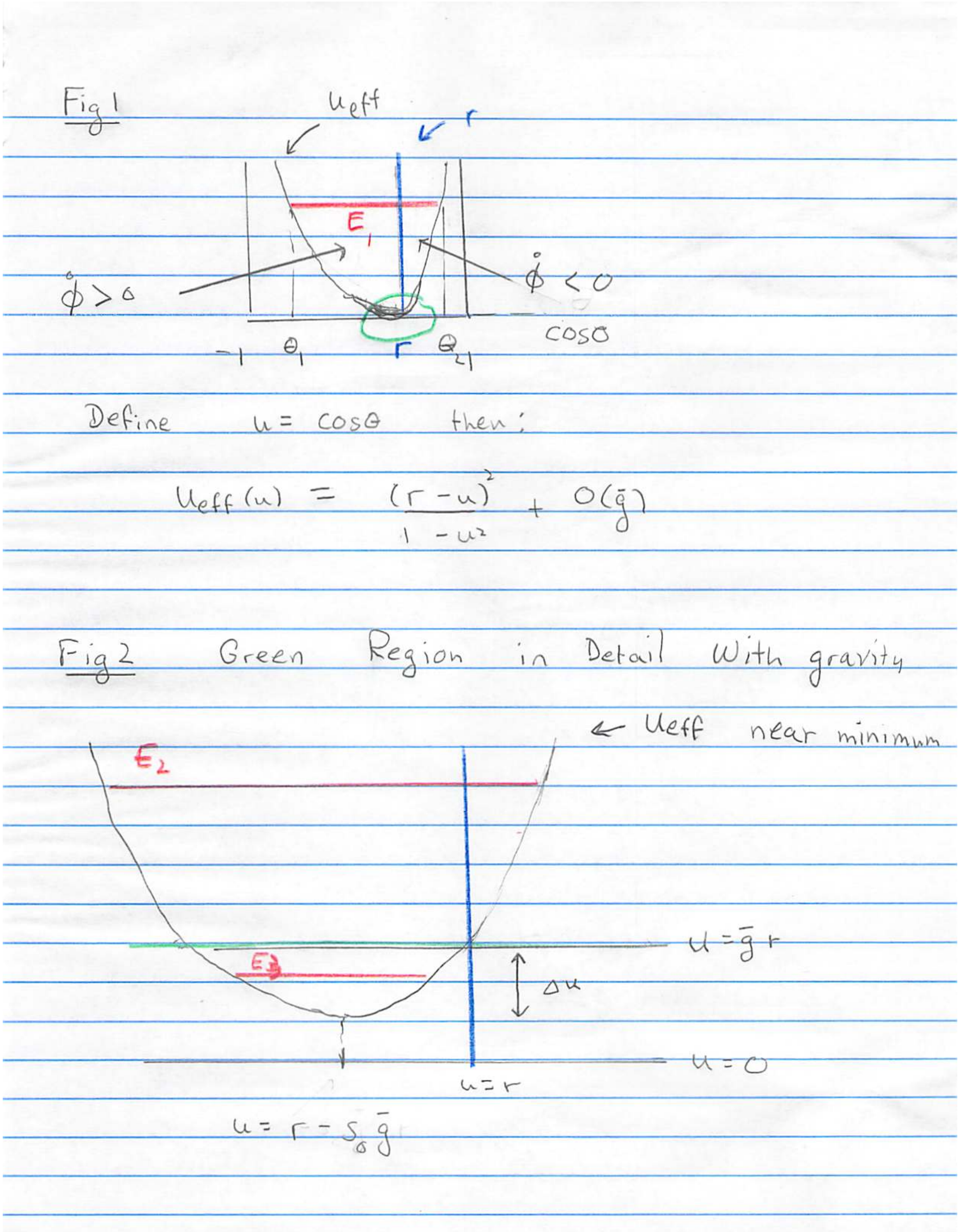


Fig 1
u_eff
r
E_1
φ̇ > 0
φ̇ < 0
cosθ
-1
θ_1
r
θ_2
1
Define u = cosθ then:
U_eff(u) = (r - u)^2 / (1 - u^2) + O(ḡ)
Fig 2 Green Region in Detail With gravity
← Ueff near minimum
E_2
u = ḡr
Δu
E_3
u = 0
u = r
u = r - S_0 ḡ

Figure 3:

Consider first the motion when in the effective potential when $E = E_1$. Then the motion will oscillate between θ_1 and θ_2 as shown in the figure Fig. 3. When $u = \cos(\theta) < r$, the precession rate $\dot{\phi}$ is positive (see Eq. (76)), while when $\cos(\theta)$ the precession rate is negative. Thus the motion in this case is the "loop-dee-loop" kind of behavior shown in Fig. 2(b). It is not possible to see the behavior seen in Fig. 2(a) without going to the next order in gravity.

Now let us include gravity. Gravity is only important when the generalized forces, $-\partial U^{(0)}_{\text{eff}}/\partial\theta$, from the zeroth order effective potential are small, i.e. near the minimum of the zero-th order effective potential. Thus we expand the green region in detail. Near this point $u \simeq r$ and we can work with $\delta u = u - r$. Then the effective potential near the minimum is approximately:

$$U_{\text{eff}}(\delta u) = \bar{g}r + \bar{g}\delta u + \frac{\delta u^2}{2(1-r^2)} \tag{77}$$

Completing the square and notating for convenience $1 - r^2 \equiv s_0$, the effective potential takes form

$$U_{\text{eff}}(\delta u) = \underbrace{\bar{g}r - \frac{1}{2}\bar{g}^2 s_0}_{\text{const}} + \frac{(\delta u + \bar{g}s_0)^2}{2s_0} \tag{78}$$

A plot of the effective potential in this region is shown by Fig. 3(b). Now if the energy is greater than $\bar{E} > \bar{g}r$ (as in $\bar{E}_2$) the motion is the "loop-dee-loop" type. If the energy $\bar{E}$ is less then $\bar{g}r$ but greater than U_{in} (as in E_3) then the motion will be of the type shown in Fig. 2(a).

To summarize we find (in the original units) the loop-dee-loop behavior is for

$$E > \frac{mg\ell I}{p_\psi}\cos\theta_0\,, \tag{79}$$

with $\cos\theta_0 = p_\phi/p_\psi$. For E in the range

$$\frac{mg\ell I}{p_\psi}\cos\theta_0 > E > \frac{mg\ell I}{p_\psi} - \frac{1}{2}\frac{(mg\ell I)^2}{p_\psi^2}\sin^2\theta_0\,. \tag{80}$$

we see the Fig. 2(a) behavior.

(c) With a clear phyical understanding we can analyze the behavior. The original Lagrangian for θ motion (technically this effective lagrangian is minus the Routhian $L_{\text{eff}} = -R$)

$$L_{\text{eff}} = \frac{1}{2}I_1\dot{\theta}^2 + U_{\text{eff}}(\theta) \tag{81}$$

Considering the discussion of the previoius item we define a variable δu .

$$\delta u = \cos(\theta) - r \tag{82}$$

When δu is positive the precession rate is negative, etc. Now

$$\delta\dot{u} = \sin(\theta)\dot{\theta} \simeq \sqrt{1-r^2}\,\dot{\theta} \tag{83}$$

So the kinetic term of the effective Lagrangian becomes

$$\frac{1}{2}I_1\dot{\theta}^2 = \frac{1}{2}\frac{I_1}{s_0}\delta\dot{u}^2 \tag{84}$$

So in our system of units

$$\frac{L_{\text{eff}}}{p_\psi^2/I_1} = \frac{1}{2}\left(\frac{I_1^2}{p_\psi^2 s_0}\right)\delta\dot{u}^2 - \frac{(\delta u + s_0\bar{g})^2}{2s_0} + \text{const} \tag{85}$$

Calculating the equation of motion of δu one finds

$$\left(\frac{I_1^2}{p_\psi^2}\right)\delta\ddot{u} = -(\delta u + s_0\bar{g}) \tag{86}$$

This is the equation of a harmonic oscillator with fequency $\omega_0 \equiv p_\psi/I_1$ oscillating around a minimum $s_0\bar{g}$. The solution

$$\delta u = -s_0\bar{g} + A\cos(\omega_0 t)\ . \tag{87}$$

So unravelling the definitions, we find $\cos\theta$ and $\dot{\phi}$ explictly as a function of time

$$\cos(\theta) - \cos(\theta_0) = -\sin^2\theta_0\frac{mg\ell I}{p_\psi^2} + A\cos(\omega_0 t) \tag{88}$$

$$\dot{\phi} = \frac{mg\ell}{p_\psi} - \frac{p_\psi}{I_1}\frac{A}{\sin^2\theta_0}\cos(\omega_0 t) \tag{89}$$

The amplitude A determines the energy of the oscillations. The top will do the loop-dee-loop if $A > \sin^2\theta_0 mg\ell I/p_\psi^2$.

Note that the time average precession rate is simply the same as one would have if one made no oscillations as derived in class

$$\bar{\dot{\phi}} = \frac{mg\ell}{p_\psi} \tag{90}$$