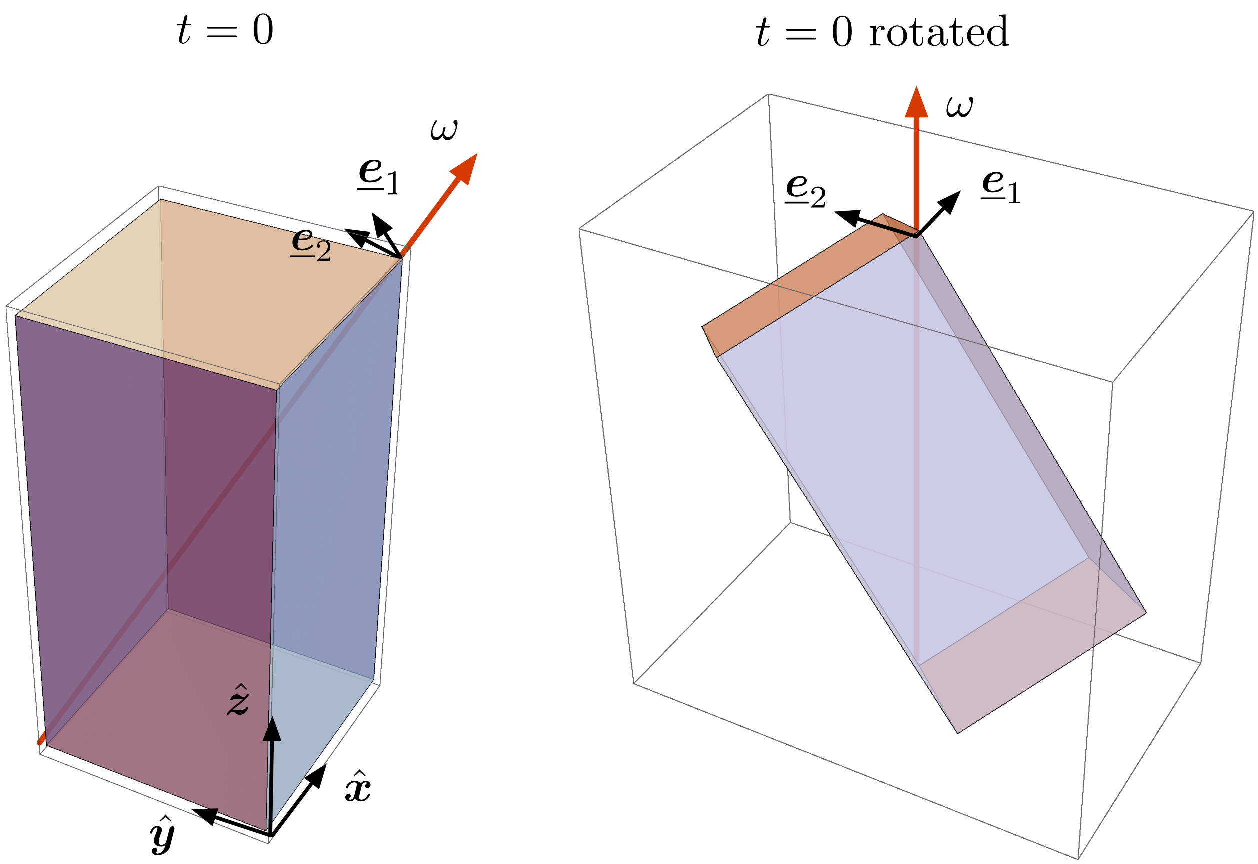
Problem 1. Torque on a box

Consider a solid box of mass m and dimension $L, L, 2L$ (see figure).



(a) Compute all components of the moment of inertia tensor around center of mass.

(b) The box is rotated with constant angular frequency ω around its diagonal. At $t = 0$ the box is oriented so that its principal axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are aligned with laboratory $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ as shown in the figure. Compute the components of angular momentum as a function of time in the body basis and in the lab basis. For the lab basis you might want use the fixed basis vectors $\underline{\boldsymbol{e}}_1$, $\underline{\boldsymbol{e}}_2$, $\underline{\boldsymbol{e}}_3$ shown in the figure, which differ by a *constant* rotation from $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.

$$\underline{\boldsymbol{e}}_1 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) \tag{1}$$

$$\underline{\boldsymbol{e}}_2 = \underline{\boldsymbol{e}}_3 \times \underline{\boldsymbol{e}}_1 \tag{2}$$

$$= \frac{1}{\sqrt{3}}(-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) \tag{3}$$

$$\underline{\boldsymbol{e}}_3 = \frac{1}{\sqrt{6}}(\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}) \tag{4}$$

(c) Compute the components of the torque required to maintain the box's rotational motion working with the rotating basis. Compute the components of the torque working with the fixed basis.

(d) (Optional) Use the Lagrangian framework to compute the required torques in the body frame.

Problem 2. (Landau) Forced oscillations the easier complex way

(a) Determine the retarded green function of the following equations:

(i)

$$\frac{da}{dt} - i\omega_0 a = 0 \tag{5}$$

(ii)

$$\ddot{x} + \eta\dot{x} = 0 \tag{6}$$

(b) Consider the driven harmonic oscillator

$$\ddot{x} + \omega_0^2 x = \frac{f(t)}{m} \tag{7}$$

Write it as an equation for $a = \dot{x} + i\omega x$, and use the Green function of (a) to find the specific solution, $a(t)$.

(c) Suppose the force approaches zero for $t \to \pm\infty$. If the oscillator was initially at rest, determine the total work done by the external force. (You should use the complex variable $a(t)$ for this calculation.)

The fourier transform of a function is defined as

$$\hat{f}(\omega) \equiv \int_{-\infty}^{\infty} e^{+i\omega t} f(t)\,. \tag{8}$$

You should find that the energy absorbed is proportional to $|\hat{f}(\omega_0)|^2$.

(d) Consider the specific force

$$f(t) = \begin{cases} F_0 & 0 < t < \tau \\ 0 & \text{otherwise} \end{cases}\,. \tag{9}$$

Determine and plot the energy in the oscillator for $t \to \infty$ as a function of $\omega_0\tau$.

Problem 3. A quick review: motion in a magnetic field

Consider a homogeneous magnetic field B_0 in the z direction, and a particle of charge q moving in three dimensions in a harmonic potential well $U = \frac{1}{2}m\omega_0^2\rho^2$, where $\rho = \sqrt{x^2+y^2}$ is the distance from the z axis.

(a) Show that for a homogeneous magnetic field the vector potential $\boldsymbol{A}$ can be written

$$\boldsymbol{A} = \frac{1}{2}B_0(-y, x, 0)\,. \tag{10}$$

(b) (Optional) Show that other ways to write the gauge field are

$$\boldsymbol{A} = B_0(-y, 0, 0)\,, \tag{11}$$

or

$$\boldsymbol{A} = B_0(0, x, 0)\,. \tag{12}$$

The choice written in part (a) is most convenient for this problem.

(c) Write down the Lagrangian for the particle in cylindrical cooridinates. It may be notationally convenient in what follows to use the cyclotron frequency

$$\omega_B \equiv \frac{qB_0}{2m}\,. \tag{13}$$

instead of the magnetic field.

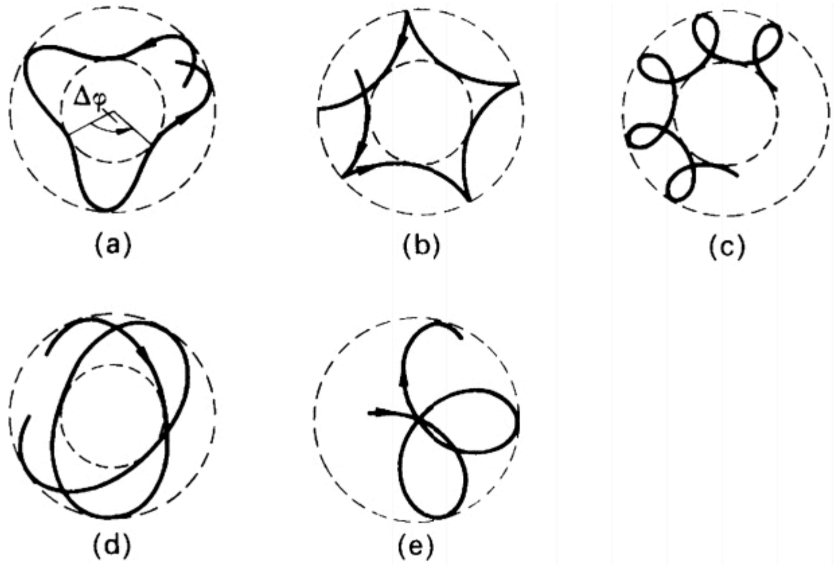
(d) Determine all conserved quantities.

(e) Show that the equation of motion for ρ takes form

$$m\ddot{\rho} = -\frac{\partial V_{\rm eff}}{\partial \rho}\,. \tag{14}$$

and determine $V_{\rm eff}$.

(f) For different values of the parameters initial conditions (or conserved quantities), the motion will be qualitatively different. Describe the range of parameters which correspond to figures (a) , (b) (c), and (d) and (e).


Δφ
(a)
(b)
(c)
(d)
(e)