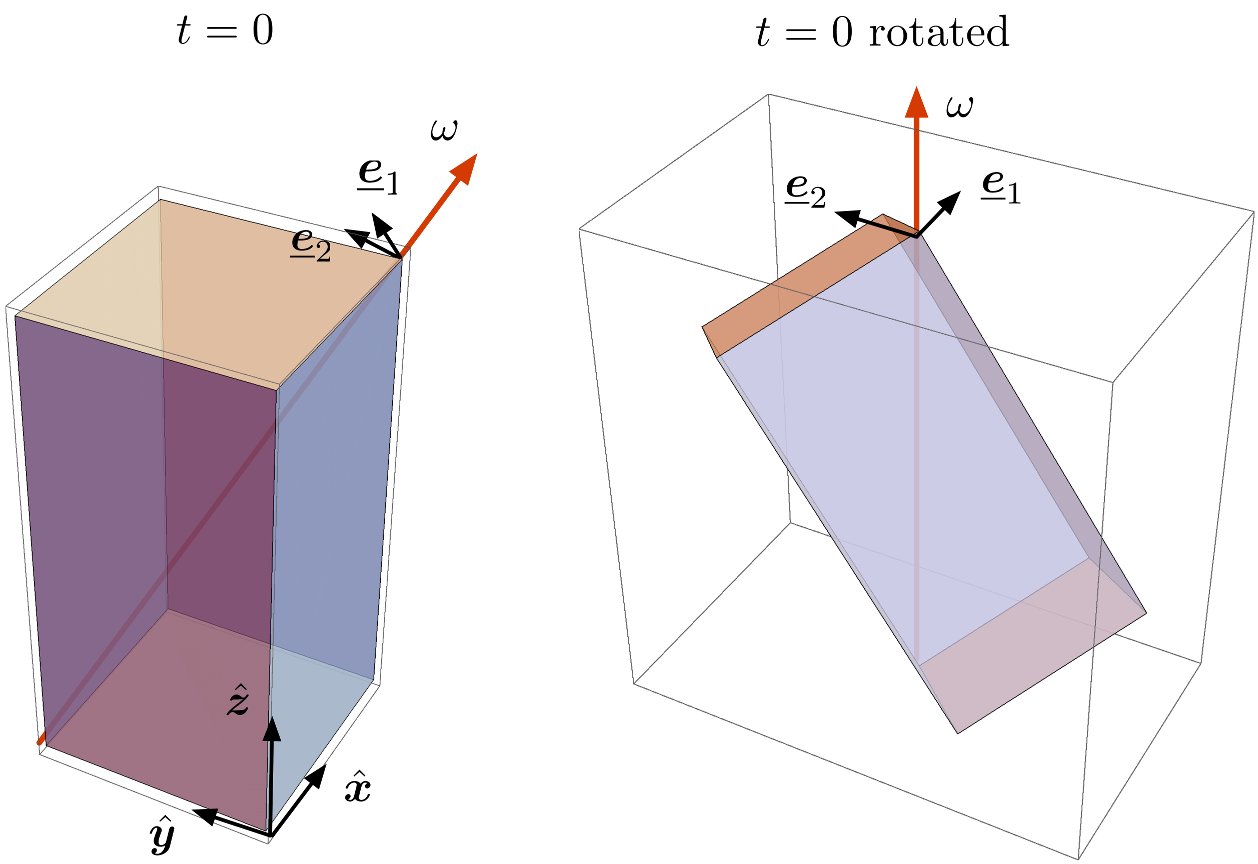
Problem 1. Torque on a box

Consider a solid box of mass m and dimension $L, L, 2L$ (see figure).



(a) Compute all components of the moment of inertia tensor around center of mass.

(b) The box is rotated with constant angular frequency ω around its diagonal. At $t = 0$ the box is oriented so that its principal axes $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$ are aligned with laboratory $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$ as shown in the figure. Compute the components of angular momentum as a function of time in the body basis and in the lab basis. For the lab basis you might want use the fixed basis vectors $\underline{\boldsymbol{e}}_1$, $\underline{\boldsymbol{e}}_2$, $\underline{\boldsymbol{e}}_3$ shown in the figure, which differ by a *constant* rotation from $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$.

$$\underline{\boldsymbol{e}}_1 = \frac{1}{\sqrt{2}}(\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}}) \tag{1}$$

$$\underline{\boldsymbol{e}}_2 = \underline{\boldsymbol{e}}_3 \times \underline{\boldsymbol{e}}_1 \tag{2}$$

$$= \frac{1}{\sqrt{3}}(-\hat{\boldsymbol{x}} + \hat{\boldsymbol{y}} + \hat{\boldsymbol{z}}) \tag{3}$$

$$\underline{\boldsymbol{e}}_3 = \frac{1}{\sqrt{6}}(\hat{\boldsymbol{x}} - \hat{\boldsymbol{y}} + 2\hat{\boldsymbol{z}}) \tag{4}$$

(c) Compute the components of the torque required to maintain the box's rotational motion working with the rotating basis. Compute the components of the torque working with the fixed basis.

(d) (Optional) Use the Lagrangian framework to compute the required torques in the body frame.

Solution:

(a) The principal axes are clearly the x, y, z coordinate system

$$I_{xx} = \int dm(y^2 + z^2) \tag{5}$$

$$I_{yy} = \int dm(x^2 + z^2) \tag{6}$$

$$I_{zz} = \int dm(x^2 + y^2) \tag{7}$$

Working through the first example

$$\begin{aligned} I_{xx} =& \frac{m}{2L^3} \times \int_{-L/2}^{L/2} dx \int_{-L/2}^{L/2} dy \int_{-L}^{L} dz(y^2 + z^2) && (8) \\ =& \frac{m}{2L^3} \left[L \times \left(\frac{2}{3} \frac{L^3}{8} \right) \times 2L + L \times L \times \frac{2}{3} L^3 \right] && (9) \\ =& mL^2 \left[\frac{1}{12} + \frac{1}{3} \right] && (10) \\ =& \frac{5}{12} mL^2 && (11) \end{aligned}$$

The other integrals work out by analogy

$$I_{yy} = I_{xx} \, , \tag{12}$$

while

$$I_{zz} = mL^2 \frac{2}{12} \, . \tag{13}$$

To summarize we have

$$I = I_0 \begin{pmatrix} 5 & & \\ & 5 & \\ & & 2 \end{pmatrix} \qquad I_0 \equiv \frac{1}{12} mL^2 \tag{14}$$

(b) **Body Frame:** Then the angular momentum components in the body axes are

$$L_a = I_{ab} \omega_b \tag{15}$$

The angular velocity components in the body axes are $(\omega_1, \omega_2, \omega_3) = \frac{\omega_0}{\sqrt{6}}(1, -1, 2)$. So we find

$$\begin{pmatrix} L_1 \\ L_2 \\ L_3 \end{pmatrix} = \frac{I_0 \omega_0}{\sqrt{6}} \begin{pmatrix} 5 & & \\ & 5 & \\ & & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix} = \frac{I_0 \omega_0}{\sqrt{6}} \begin{pmatrix} 5 \\ -5 \\ 4 \end{pmatrix} \tag{16}$$

Lab frame components physical description: First consider $t = 0$. Then at this time then vector the body axes $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are aligned with $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$:

$$\boldsymbol{L}\big|_{t=0} = L_1 \boldsymbol{e}_1 + L_2 \boldsymbol{e}_2 + L_3 \boldsymbol{e}_3 = L_1 \hat{\mathbf{x}} + L_2 \hat{\mathbf{y}} + L_3 \hat{\mathbf{z}} \tag{17}$$

Given the relation between the $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ vectors and $\underline{\boldsymbol{e}}_1, \underline{\boldsymbol{e}}_2, \underline{\boldsymbol{e}}_3$

$$\begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{e}}_1 \\ \underline{\boldsymbol{e}}_2 \\ \underline{\boldsymbol{e}}_3 \end{pmatrix} \tag{18}$$

The matrix here is the transpose of the relations given in Eq. (1)

$$\begin{pmatrix} \underline{\boldsymbol{e}}_1 \\ \underline{\boldsymbol{e}}_2 \\ \underline{\boldsymbol{e}}_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} \hat{\mathbf{x}} \\ \hat{\mathbf{y}} \\ \hat{\mathbf{z}} \end{pmatrix} \tag{19}$$

It is straightforward to show that

$$\boldsymbol{L}\big|_{t=0} = I_0\omega\left(-\sqrt{2}\underline{\boldsymbol{e}}_2 + 3\underline{\boldsymbol{e}}_3\right) \tag{20}$$

At a later time $\boldsymbol{L}$ simply precesses around the $\underline{\boldsymbol{e}}_3$ axis

$$\underline{\boldsymbol{e}}_2 \to \sin(\omega t)\underline{\boldsymbol{e}}_1 - \cos(\omega t)\underline{\boldsymbol{e}}_2 \tag{21}$$

leading to

$$\boldsymbol{L}(t) = I_0\omega\left(\sqrt{2}\sin(\omega t)\underline{\boldsymbol{e}}_1 - \sqrt{2}\cos(\omega t)\underline{\boldsymbol{e}}_2 + 3\underline{\boldsymbol{e}}_3\right) \tag{22}$$

Lab frame components mathematical description: The angular momentum is

$$\boldsymbol{L} = L_a\boldsymbol{e}_a(t) = L_{\underline{a}}(t)\underline{\boldsymbol{e}}_a \tag{23}$$

So taking the dot product with $\underline{\boldsymbol{e}}_b$ we have

$$L_{\underline{b}}(t) = L_a\boldsymbol{e}_a(t)\cdot\underline{\boldsymbol{e}}_b \tag{24}$$

$$= L_aR_{ab}(t) \tag{25}$$

where

$$\boldsymbol{e}_a = R_{ab}\underline{e}_b \tag{26}$$

R_{ab} takes the fixed basis $\underline{\boldsymbol{e}}_a$ to the principal axes $\boldsymbol{e}_a$.

R_{ab} is conveniently expressed as a sequence of rotations parametrized by Euler angles, ϕ, θ, ψ. First there is a rotation around the z axis

$$R_1 \equiv \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

At $t = 0$, we have $\phi = 0$. Later $\phi = \omega t$. Then a rotation around the new x axis by angle $-\theta$ which is the angle between the $\hat{z}$ axis and ω

$$\cos\theta = \hat{\mathbf{z}}\cdot\hat{\boldsymbol{\omega}} = \frac{2}{\sqrt{6}} \tag{28}$$

$$\sin\theta = 1/\sqrt{3} \tag{29}$$

Here

$$R_2 \equiv R_x(-\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{3}} \\ 0 & \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} \end{pmatrix} \tag{30}$$

Finally we have a rotation by $\psi = -\pi/4$ around the z-axis around the z axis

$$R_3 \equiv R_z(-\pi/4) = \begin{pmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{31}$$

The matrix then is

$$R = R_z(-\pi/4)R_x(-\theta)R_z(\omega t) \tag{32}$$

So we have

$$\begin{pmatrix} L_{\underline{1}} & L_{\underline{2}} & L_{\underline{3}} \end{pmatrix} = \begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix} \begin{pmatrix} & R_3 \cdot R_2 \cdot R_1 & \end{pmatrix} \tag{33}$$

Only the last matrix is time dependent. Multiplying out these expression we have

$$\boldsymbol{L} = I_0\omega \left[\sqrt{2}\sin(\omega t)\underline{\boldsymbol{e}}_1 - \sqrt{2}\cos(\omega t)\underline{\boldsymbol{e}}_2 + 3\underline{\boldsymbol{e}}_3 \right] \tag{34}$$

This can also be obtained from geometrical considerations as done above.

(c) **Body frame:** The torque in the body frame is

$$\begin{aligned} \boldsymbol{\tau} =& \frac{d\boldsymbol{L}}{dt} && (35) \\ =& \frac{dL_a}{dt}\boldsymbol{e}_a + \boldsymbol{\omega} \times \boldsymbol{L} && (36) \\ =& \boldsymbol{\omega} \times \boldsymbol{L} && (37) \\ =& \frac{\omega}{\sqrt{6}}(\boldsymbol{e}_1 - \boldsymbol{e}_2 + 2\boldsymbol{e}_3) \times \frac{I_0\omega}{\sqrt{6}}(5\boldsymbol{e}_1 - 5\boldsymbol{e}_2 + 4\boldsymbol{e}_3) && (38) \\ =& I_0\omega^2(\boldsymbol{e}_1 + \boldsymbol{e}_2) && (39) \end{aligned}$$

Lab Frame: Here we differentiate Eq. (34)

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = I_0\omega^2 \left[\sqrt{2}\cos(\omega t)\underline{\boldsymbol{e}}_1 + \sqrt{2}\sin(\omega t)\underline{\boldsymbol{e}}_2 \right] \tag{40}$$

The connection between the Lab Frame and body frame formulas can be realized by examining the $t = 0$ case where we see clearly that

$$\boldsymbol{e}_1(t) + \boldsymbol{e}_2(t)\big|_{t=0} = \sqrt{2}\underline{\boldsymbol{e}}_1 \tag{41}$$

At a later time $\boldsymbol{e}_1(t) + \boldsymbol{e}_2(t)$ is simply rotated by an angle ϕ around the $\underline{\boldsymbol{e}}_3$ axis leaving

$$\boldsymbol{e}_1(t) + \boldsymbol{e}_2(t) = \sqrt{2}\left(\cos(\omega t)\underline{\boldsymbol{e}}_1 + \sin(\omega t)\underline{\boldsymbol{e}}_2\right) \tag{42}$$

Problem 2. (Landau) Forced oscillations the easier complex way

(a) Determine the retarded green function of the following equations:

(i)

$$\frac{da}{dt} - i\omega_0 a = 0 \tag{43}$$

(ii)

$$\ddot{x} + \eta\dot{x} = 0 \tag{44}$$

(b) Consider the driven harmonic oscillator

$$\ddot{x} + \omega_0^2 x = \frac{f(t)}{m} \tag{45}$$

Write it as an equation for $a = \dot{x} + i\omega x$, and use the Green function of (a) to find the specific solution, $a(t)$.

(c) Suppose the force approaches zero for $t \to \pm\infty$. If the oscillator was initially at rest, determine the total work done by the external force. (You should use the complex variable $a(t)$ for this calculation.)

The fourier transform of a function is defined as

$$\hat{f}(\omega) \equiv \int_{-\infty}^{\infty} e^{+i\omega t} f(t)\,. \tag{46}$$

You should find that the energy absorbed is proportional to $|\hat{f}(\omega_0)|^2$.

(d) Consider the specific force

$$f(t) = \begin{cases} F_0 & 0 < t < \tau \\ 0 & \text{otherwise} \end{cases}\,. \tag{47}$$

Determine and plot the energy in the oscillator for $t \to \infty$ as a function of $\omega_0\tau$.

(a) (i) For the first equation we try to solve

$$\left(\frac{d}{dt} - i\omega\right) G(t, t_0) = \delta(t - t_0) \tag{48}$$

It is a first order differential equation. For $t > t_0$ the general solution is

$$G(t, t_0) = Ae^{i\omega t} \tag{49}$$

For $t < t_0$ the retarded Green function is zero:

$$G(t, t_0) = 0 \tag{50}$$

Then integrating Eq. (48) from $t = t_0 - \epsilon$ to $t_0 + \epsilon$ gives

$$G(t_0 + \epsilon, t_0) - \underbrace{G(t_0 - \epsilon, t_0)}_{=0} = 1 \tag{51}$$

So we may adjust A so that this (Eq. (51)) is satisfied yielding

$$G(t, t_0) = \theta(t - t_0)e^{i\omega_0(t-t_0)} \tag{52}$$

(ii) For the second equation we solve for $t > t_0$ and find

$$G(t, t_0) = A + Be^{-\eta t} \tag{53}$$

while for $t < 0$ the green function is zero. Demanding continuity at $t = 0$ of these two solutions we find we find

$$G(t, t_0) = A(1 - e^{-\eta(t-t_0)}) \tag{54}$$

To determine the remaining constant we integrate from $t_0 - \epsilon$ to $t_0 + \epsilon$ yielding

$$\int_{t_0-\epsilon}^{t_0+\epsilon} \left(\frac{d^2G}{dt^2} + \eta\frac{d}{dt}G\right) = \int_{t_0-\epsilon}^{t_0+\epsilon} \delta(t - t_0) \tag{55}$$

yielding

$$\partial_t G(t, t_0)|_{t_0-\epsilon}^{t_0+\epsilon} + \eta G(t, t_0)|_{t_0-\epsilon}^{t_0+\epsilon} = 1 \tag{56}$$

So since $G \to 0$ as $t \to t_0$ we find

$$\partial_t G(t, t_0) = 1\,. \tag{57}$$

This fixes the coefficient of A in Eq. (54) establishing that

$$G(t, t_0) = \frac{1}{\eta}(1 - e^{-\eta(t-t_0)})\theta(t - t_0)\,. \tag{58}$$

(b) We write

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{d}{dt}(\dot{x} + i\omega x) - i\omega(\dot{x} + i\omega x) \tag{59}$$

Thus the equation of motion is

$$\frac{da}{dt} - i\omega_o a = \frac{f(t)}{m}\,, \tag{60}$$

Using the Green function we find

$$a(t) = \int_{-\infty}^{\infty} dt_0\, f(t_0) G(t, t_0) \tag{61}$$

$$= e^{i\omega_0 t} \int_{-\infty}^{t} dt_0\, f(t_0) e^{-i\omega_0 t_0} \tag{62}$$

(c) For the specific force we can integrate

$$a(t) = e^{i\omega_0 t} \frac{F_0}{m} \int_0^{\tau} dt_0 e^{-i\omega_0 t_0} \tag{63}$$

yielding

$$a(t) = e^{i\omega_0 t} \frac{F}{-i\omega_0 m}(1 - e^{-i\omega_0 \tau}) \tag{64}$$

The energy is

$$E(t,\tau) = \frac{1}{2} m\dot{x}^2 + \frac{1}{2} m\omega_0^2 x^2 \tag{65}$$

$$= \frac{m}{2} |a(t,\tau)|^2 \tag{66}$$

$$= \frac{2F_0^2}{m\omega_0^2} \sin^2(\omega_0 \tau/2) \tag{67}$$

In the limit of a short force $\omega_0 \tau \ll 1$ the impulse is $F_0 \tau$. So, the velocity after the impulse is $v = (F_0 \tau)/m$. And the energy in the oscillator just after the impulse is $1/2mv^2 = (F_0\tau)^2/2m$. Expanding our expression in Eq. (67) for $\omega_0 \tau \ll 1$, it gracefully approaches $(F_0\tau)^2/2m$.

Problem 3. A quick review: motion in a magnetic field

Consider a homogeneous magnetic field B_0 in the z direction, and a particle of charge q moving in three dimensions in a harmonic potential well $U = \frac{1}{2}m\omega_0^2\rho^2$, where $\rho = \sqrt{x^2+y^2}$ is the distance from the z axis.

(a) Show that for a homogeneous magnetic field the vector potential $\boldsymbol{A}$ can be written

$$\boldsymbol{A} = \frac{1}{2}B_0(-y, x, 0)\,. \tag{68}$$

(b) (Optional) Show that other ways to write the gauge field are

$$\boldsymbol{A} = B_0(-y, 0, 0)\,, \tag{69}$$

or

$$\boldsymbol{A} = B_0(0, x, 0)\,. \tag{70}$$

The choice written in part (a) is most convenient for this problem.

(c) Write down the Lagrangian for the particle in cylindrical cooridinates. It may be notationally convenient in what follows to use the cyclotron frequency

$$\omega_B \equiv \frac{qB_0}{2m}\,. \tag{71}$$

instead of the magnetic field.

(d) Determine all conserved quantities.

(e) Show that the equation of motion for ρ takes form

$$m\ddot{\rho} = -\frac{\partial V_{\rm eff}}{\partial \rho}\,. \tag{72}$$

and determine $V_{\rm eff}$.

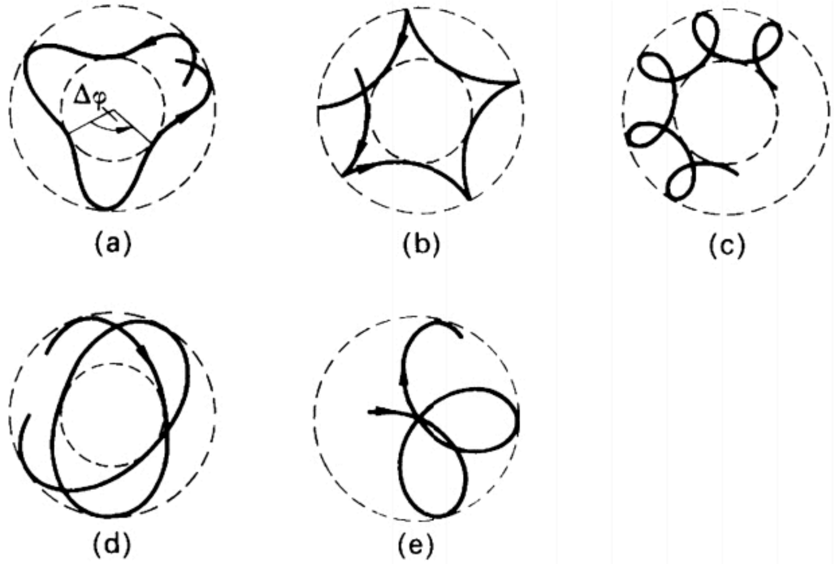
(f) For different values of the parameters initial conditions (or conserved quantities), the motion will be qualitatively different. Describe the range of parameters which correspond to figures (a) , (b) (c), and (d) and (e).

Show, for instance, that case (c) is when $p_\phi > 0$ and

$$E_\perp > p_\phi \frac{\omega_0^2}{2\omega_B^2} \tag{73}$$

with $E_\perp = E - p_z^2/2m$.

Hint: Consider three cases $p_\phi > 0$, $p_\phi < 0$, and $p_\phi = 0$. Pay attention to the arrows in the figures.


Δφ
(a)
(b)
(c)
(d)
(e)

(a) Just differentiate:

$$B_z = \partial_x A_y - \partial_y A_x = B_0 \tag{74}$$

(b) These are the same as part a. The two different forms for $\boldsymbol{A}$ differ by the gradient of a scalar function

$$B_0(-y,0,0) = \frac{1}{2}B_0(-y,x,0) - \nabla\Lambda \tag{75}$$

with $\Lambda = \frac{1}{2}B_0 xy$

(c) We have

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{q}{c}\boldsymbol{v}\cdot\boldsymbol{A} - \frac{1}{2}m\omega_0^2\rho^2 \tag{76}$$

$$= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \frac{qB_0}{2c}(-\dot{x}y + \dot{y}x) - \frac{1}{2}m\omega_0^2\rho^2 \tag{77}$$

Now one can recognize $x\dot{y} - y\dot{x}$ as being proportional to the angular momentum

$$(-\dot{x}y + \dot{y}x) = \rho^2\dot{\phi} \tag{78}$$

So the final Lagrangian is

$$L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2\dot{z}^2) + m\omega_B\rho^2\dot{\phi} - \frac{1}{2}m\omega_0^2\rho^2 \tag{79}$$

(d) The momentum is conserved

$$p_z \tag{80}$$

The conserved quantities are the angular momentum

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = m\rho^2\dot{\phi} + m\rho^2\omega_B \tag{81}$$

We note that

$$\dot{\phi} = \frac{p_\phi}{m\rho^2} - \omega_B \tag{82}$$

The energy is conserved

$$h = p_\phi\dot{\phi} + p_\rho\dot{\rho} - L \tag{83}$$

$$= \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 +) + \frac{1}{2}m\omega_0^2\rho^2 \tag{84}$$

We note that the energy is

$$h = \frac{1}{2}m\dot{\rho}^2 + \frac{(p_\phi - m\rho^2\omega_B)^2}{2m\rho^2} + \frac{1}{2}m\omega_0^2\rho^2 \tag{85}$$

$$= \frac{1}{2}m\dot{\rho}^2 - p_\phi\omega_B + \left(\frac{p_\phi^2}{2m\rho^2} + \frac{1}{2}m(\omega_0^2 + \omega_B^2)\rho^2\right) \tag{86}$$

$$= \frac{1}{2}m\dot{\rho}^2 - p_\phi\omega_B + V_{\text{eff}}(\rho) \tag{87}$$

(e) The equation for ρ is

$$m\ddot{\rho} = m\rho\dot{\phi}^2 + 2m\omega_B\rho\dot{\phi} - \partial_\rho U(\rho) \tag{88}$$

with $U = \frac{1}{2}m\omega_0^2\rho^2$. Substituting Eq. (82) we find

$$m\ddot{\rho} = \frac{p_\phi^2}{m\rho^3} - m\omega_B^2\rho - \partial_\rho U(\rho) \tag{89}$$

$$= -\frac{V_{\rm eff}}{\partial\rho} \tag{90}$$

where

$$V_{\rm eff} = \frac{p_\phi^2}{2m\rho^2} + \frac{1}{2}m(\omega_0^2 + \omega_B^2)\rho^2 \tag{91}$$

(f) We first define

$$E_\perp = E - \frac{p_z^2}{2m} \tag{92}$$

and

$$\mathcal{E} = E_\perp + p_\phi\omega_B \tag{93}$$

The motion has

$$\mathcal{E} = \frac{1}{2}m\dot{\rho}^2 + V_{\rm eff}(\rho) \tag{94}$$

If p_ϕ is negative then

$$\dot{\phi} < 0 \tag{95}$$

This is the case in (d) .

If p_ϕ is positive $\dot{\phi}$ could be positive or negative depending on ρ

$$\dot{\phi} = \frac{p_\phi}{m\rho^2} - \omega_B \tag{96}$$

It is helpful here to set $m = 1$ and to define

$$u = \frac{1}{\rho^2} . \tag{97}$$

Thus

$$\dot{\phi} = p_\phi u - \omega_B \tag{98}$$

So, the system will be switch from $\dot{\phi}$ positive to negative for u less than

$$u < u_{\rm crit} = \frac{\omega_B}{p_\phi} \tag{99}$$

or

$$\rho > \sqrt{\frac{p_\phi}{\omega_B}} \tag{100}$$

In order to reach this value of ρ the energy needs to be sufficiently large.

The turning points in "u" space is found by setting $\dot{\rho} = 0$

$$\mathcal{E} = \frac{p_\phi^2}{2} u + \frac{1}{2} \frac{(\omega_0^2 + \omega_B^2)}{u} \tag{101}$$

The two roots of this equation are

$$u_\pm = \frac{\mathcal{E} \pm \sqrt{\mathcal{E}^2 - p_\phi^2(\omega_B^2 + \omega_0^2)}}{p_\phi^2} \tag{102}$$

We immediately conclude that the lowest possible $\mathcal{E}$ (which corresponds to a circular orbit where $u_+ = u_-$) is

$$E_\perp^{\text{min}} = p_\phi \sqrt{\omega_B^2 + \omega_0^2} - p_\phi \omega_B \tag{103}$$

The relevant question to ask is wether u_- (the greatest ρ) satisfies $u_- < \omega_B / p_\phi$. If this is the case we will have $\dot{\phi}$ negative corresponding to case (c). Setting

$$\frac{\mathcal{E} - \sqrt{\mathcal{E}^2 - p_\phi^2(\omega_B^2 + \omega_0^2)}}{p_\phi^2} = \frac{\omega_B}{p_\phi} \tag{104}$$

and solving for the energy we see that

$$\mathcal{E}_{\text{crit}} = p_\phi \omega_B \left(1 + \frac{\omega_0^2}{2\omega_B^2}\right) \tag{105}$$

So we have

$$E_\perp^{\text{crit}} = \frac{p_\phi \omega_0^2}{2\omega_B} \tag{106}$$

To summarize we have

$$E_\perp^{\text{min}} < E < E_\perp^{\text{crit}} \tag{107}$$

Then we have case (a). Case (b) is $E = E_\perp^{\text{crit}}$ and (c) is $E > E_\perp^{\text{crit}}$.

Finally we note that case (e) the particle goes right through the center. Looking at the effective potential at small ρ, we see that

$$V_{\text{eff}} \propto \frac{p_\phi^2}{2m\rho^2} \tag{108}$$

Thus it is only possible to pass through the origin with finite energy if $p_\phi = 0$.