Problem 1. Brownian motion of a Harmonic Oscillator

In equilibrium at temperature[1] T, the probability distribution for a harmonic oscillator to have velocity v and position x is

$$P(x,v) = Ce^{-E(x,v)/T} dx\, dv\,, \tag{1}$$

where

$$E(x,v) = \frac{1}{2}mv^2 + \frac{1}{2}mx^2\,, \tag{2}$$

and C is a normalizing constant. In what follows $\langle U \rangle$ denotes and average with respect to the probability $P(x,v)$

(a) Show that

$$\langle E \rangle = k_B T\,, \tag{3}$$

by integrating over the probability distribution.

The oscilllator is coupled to the a medium. The medium supplies random forces[2] $f(t)$ and causes dissipation with drag coefficient η, so that the equation motion that the harmonic oscillator satisfies is

$$m\frac{d^2x}{dt^2} + m\eta\frac{dx}{dt} + m\omega_0^2 = f(t)\,. \tag{4}$$

(a) Write down a formal solution for $x(t)$ assuming that it starts from rest at position $x = 0$. Use this to find the energy $E(t)$ of the oscillator as a function of time.

The result for the solution is a single integral over time t_1 with $f(t_1)$, while the result for the energy is a double integral over t_1 and t_2 with $f(t_1)$ and $f(t_2)$.

(b) Now assume that the force is random and only correlated over a short period of time[3]

$$\overline{f(t)} = 0\,, \tag{5a}$$

$$\overline{f(t)f(t')} = \kappa\delta(t-t')\,, \tag{5b}$$

with κ an aribitrary constant. Show that with the statistics in Eq. (5) the average energy as function of time is

$$\overline{E}(t) = \frac{\kappa}{2m\eta}(1 - e^{-\eta t})\,. \tag{6}$$

(c) In order to reach the correct equilibrium given in Eq. (3) at late times what must κ be? This establishes a fundamental connection between the dissipation, as quantified by the drag coefficient η, and the noise, as quantified by the force coefficient κ.

[1] I have and will continue to set $k_B = 1$ so T in this problem is short for $k_B T$.

[2] We mean that the time average of the forces is zero, $\overline{f} = \int_{-\tau}^{\tau} dt f(t) = 0$. Here τ is long compared to the microscopic time scales of the medium, but short compared to the time scales of the oscillator, $2\pi/\omega_0$.

[3] $\overline{U}$ denotes an average over time, and is a priori independent of $\langle U \rangle$.