Problem 1. A non-linear oscillator

An oscillator of mass m and resonant frequency ω_0 has a damping force $F_D = -\beta v^3$ with $\beta > 0$. The motion is initialized with amplitude a_0 and no velocity at time $t = 0$.

(a) Define suitable dimensionless variables so that a dimensionless version of the equation reads:

$$\frac{d^2\overline{x}}{d\overline{t}^2} + \overline{x} + \epsilon \left(\frac{d\bar{x}}{d\bar{t}} \right)^3 = 0 \tag{1}$$

What is the condition on β that the non-linear term may be considered small?

(b) If the oscillator starts at $\bar{t} = 0$ with $\bar{x} = 1$ with $d\bar{x}/d\bar{t} = 0$, use secular perturbation theory to determine approximate behavior of $\bar{x}(\bar{t})$. Show in particular that the amplitude decreases as $\bar{t}^{-1/2}$ at late times.

(c) (Optional) Use Mathematica or other program to determine the exact numerical solution[1], and plot the exact and approximate solution for $\epsilon = 0.3$ up to a time $\bar{t} = 160$. The picture I get is shown below.

[1] Look up `NDSolve` and figure it out. I find the following Mathematica advice (parts I and II) by my friend and colleague Mark Alford useful.

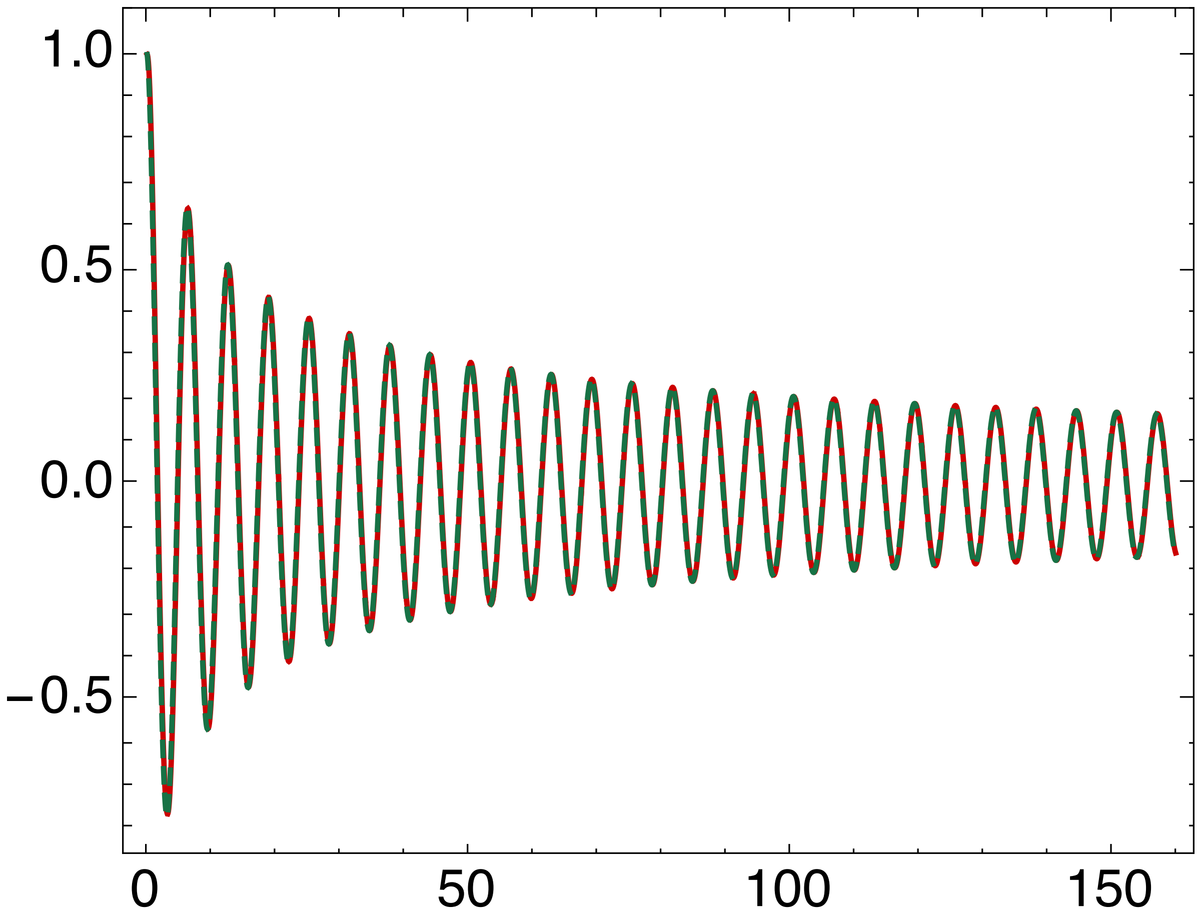

Figure 1:

Problem 2. (Likharev) An effective mass

For a system with the following Lagrangian: Consider an oscillator with generalized coordinate q and resonant frequency ω_0 with an effective mass which is weakly coordinate-dependent $m_{\rm eff} = m(1 + \epsilon q^2)$ where m is a constant[2] . The lagrangian is

$$L = \frac{1}{2} m_{\rm eff}(q)\dot{q}^2 - \frac{1}{2} m\omega_0^2 q^2 \tag{2}$$

Calculate the frequency of oscillations using secular perturbation theory, and from an integral given in class for the period of one dimensional systems (see "Motion of 1d systems" online). Assume that the amplitude of the oscillations is A and that $\epsilon A^2 \ll 1$.

You should find by both methods that to first order in ϵA^2

$$\omega \simeq \omega_0 \left(1 - \frac{A^2\epsilon}{4}\right) . \tag{3}$$

Give a qualitative explanation for why the shift is negative.

[2]Technically m has units, $[m] = {\rm kg} \cdot {\rm m}^2$ and q is dimensionless

Problem 3. Anharmonic oscillations to quadratic order

Consider the oscillator with energy E in the potential

$$U = \frac{1}{2}m\omega_0^2 q^2 + \frac{c}{3}q^3 \tag{4}$$

where the anharmonic contribution is small. The oscillator is at the top of its arc at $t = 0$. We will determine an approximation to $q(t)$

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} \tag{5}$$

to second order in c.

(a) Choose an appropriate set of units so that the equation of motion can be written with

$$\frac{d^2\bar{q}}{d\bar{t}^2} + \bar{q} + \bar{c}\,\bar{q}^2 = 0\,, \tag{6}$$

with initial condition $\bar{q}(0) = 1$. $\bar{q}$, $\bar{c}$ and $\bar{t}$ are dimensionless versions of q, c and t. To lighten the notation we will drop the bars for the remainder of this problem. $\bar{c}$ is small in this problem; what does this imply for c?

(b) Solve for $q^{(0)}$, $q^{(1)}$, and $q^{(2)}$. You should find to order c^2

$$q(t) = a\cos(\omega t) - \frac{a^2 c}{2} + \frac{a^2 c}{6}\cos(2\omega t) + \frac{a^3 c^2}{48}\cos(3\omega t) \tag{7}$$

with

$$\omega = 1 - \frac{5c^2}{12} + \ldots \tag{8}$$

and amplitude a adjusted to reproduce the initial condition $q(0) = 1$:

$$1 = a - \frac{a^2}{2}c + \frac{a^2 c}{6} + \frac{a^3 c^2}{48} \tag{9}$$

or

$$a(c) = 1 + \frac{1}{3}c + \frac{29}{144}c^2 + \ldots \tag{10}$$

Hint: A general approach is to try a zero-th order ansatz of the form

$$q^{(0)} = A(t)\cos(-\omega_0 t + \varphi(t)) \tag{11}$$

and this will always work. Here you could try a more restricted form, by making an anasatz for $A(t)$ and an ansatz for $\varphi(t)$, and write the zeroth order solution as[3]

$$q^{(0)} = A\cos(\omega t) \tag{12}$$

[3] In this ansatz we treat $A(t)$ as a constant, and write $\varphi(t) = -\Delta\omega t$. So the frequency is $\omega = \omega_0 + \Delta\omega$.

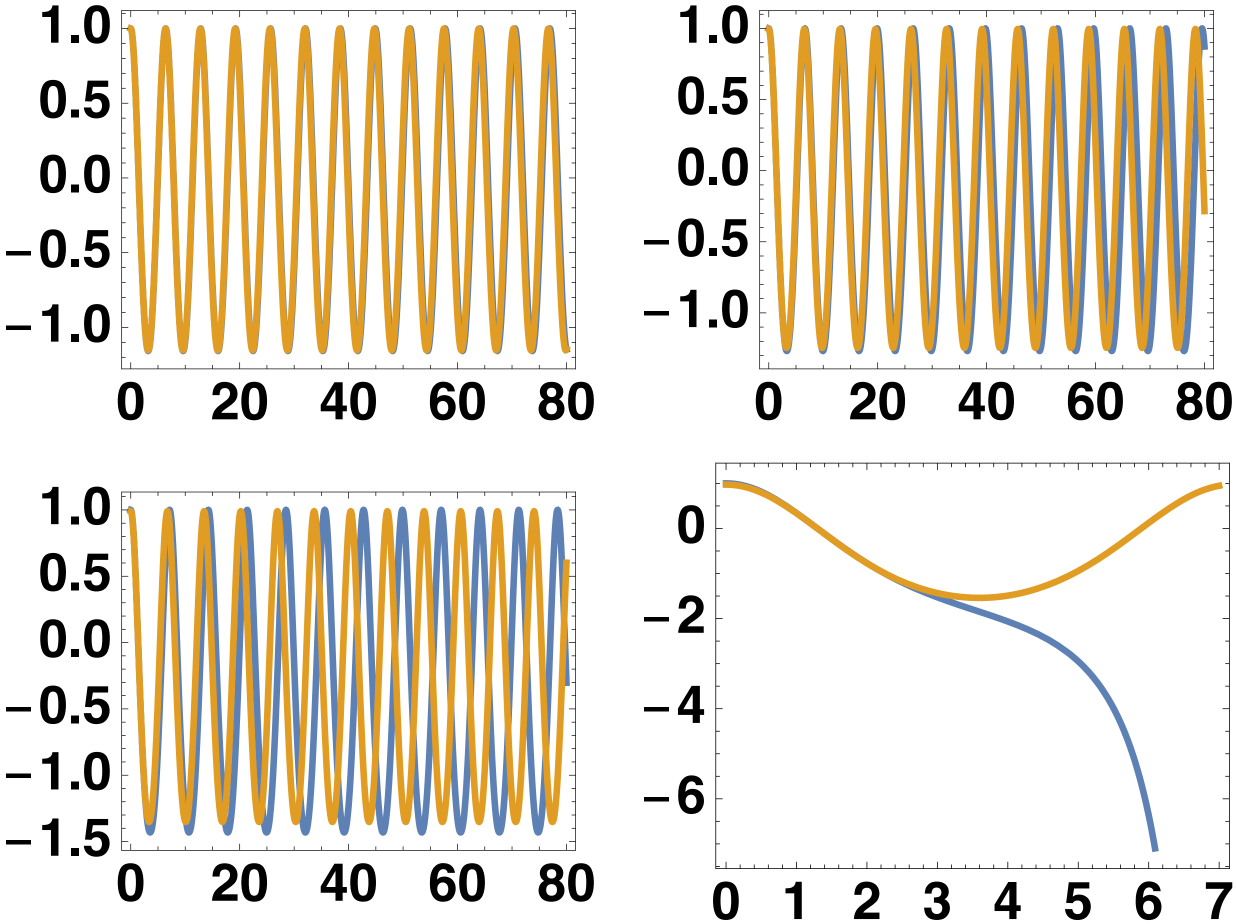

Figure 2: Analytical perturbative solution (yellow curve) compared to the numerical solution (blue curve) versus time for $q(t)$. Reading the graphs like words in a book, the comparison is for $c = 0.2, 0.3, 0.4$ and 0.55 (so $c = 0.3$ is the top right graph).

where A and ω are adjusted order by order to remove secular terms:

$$A = 1 + \lambda A_1 + \lambda^2 A_2 + \dots \tag{13}$$

$$\omega = 1 + \lambda\omega_1 + \lambda^2\omega_2 + \dots \tag{14}$$

(Here λ is a book keeping parameter which denotes the order in $\bar{c}$, i.e. λ^2 denotes a term of order $\bar{c}^2$. We will set $\lambda = 1$ at the end.) One could hope that this ansatz will work here since the energy is fixed, and the period of the non-linear oscillations is also fixed. If the ansatz in Eq. (12) was not general enough, then there will be secular terms which can't be captured with this form, and then you would go back to the more general approach in Eq. (11).

(c) The graph in Fig 2 compares solution in Eq. (7) to a numerical solution. Explain why the perturbative solution fails qualitatively for $c = 0.55$.

(d) The motion is periodic with period T. Qualitatively sketch the power spectrum, i.e. if $q(t)$ is expanded in a Fourier series, $q(t) = \sum_n q_n e^{-i2\pi nt/T}$, sketch $|q_n|^2$ versus n. How does increasing the non-linearity c change this spectrum?