Problem 1. A non-linear oscillator

An oscillator of mass m and resonant frequency ω_0 has a damping force $F_D = -\beta v^3$ with $\beta > 0$. The motion is initialized with amplitude a_0 and no velocity at time $t = 0$.

(a) Define suitable dimensionless variables so that a dimensionless version of the equation reads:

$$\frac{d^2\overline{x}}{d\overline{t}^2} + \overline{x} + \epsilon \left(\frac{d\bar{x}}{d\bar{t}}\right)^3 = 0 \tag{1}$$

What is the condition on β that the non-linear term may be considered small?

(b) If the oscillator starts at $\bar{t} = 0$ with $\bar{x} = 1$ with $d\bar{x}/d\bar{t} = 0$, use secular perturbation theory to determine approximate behavior of $\bar{x}(\bar{t})$. Show in particular that the amplitude decreases as $\bar{t}^{-1/2}$ at late times.

(c) (Optional) Use Mathematica or other program to determine the exact numerical solution[1], and plot the exact and approximate solution for $\epsilon = 0.3$ up to a time $\bar{t} = 160$. The picture I get is shown below.

[1] Look up `NDSolve` and figure it out. I find the following Mathematica advice (parts I and II) by my friend and colleague Mark Alford useful.

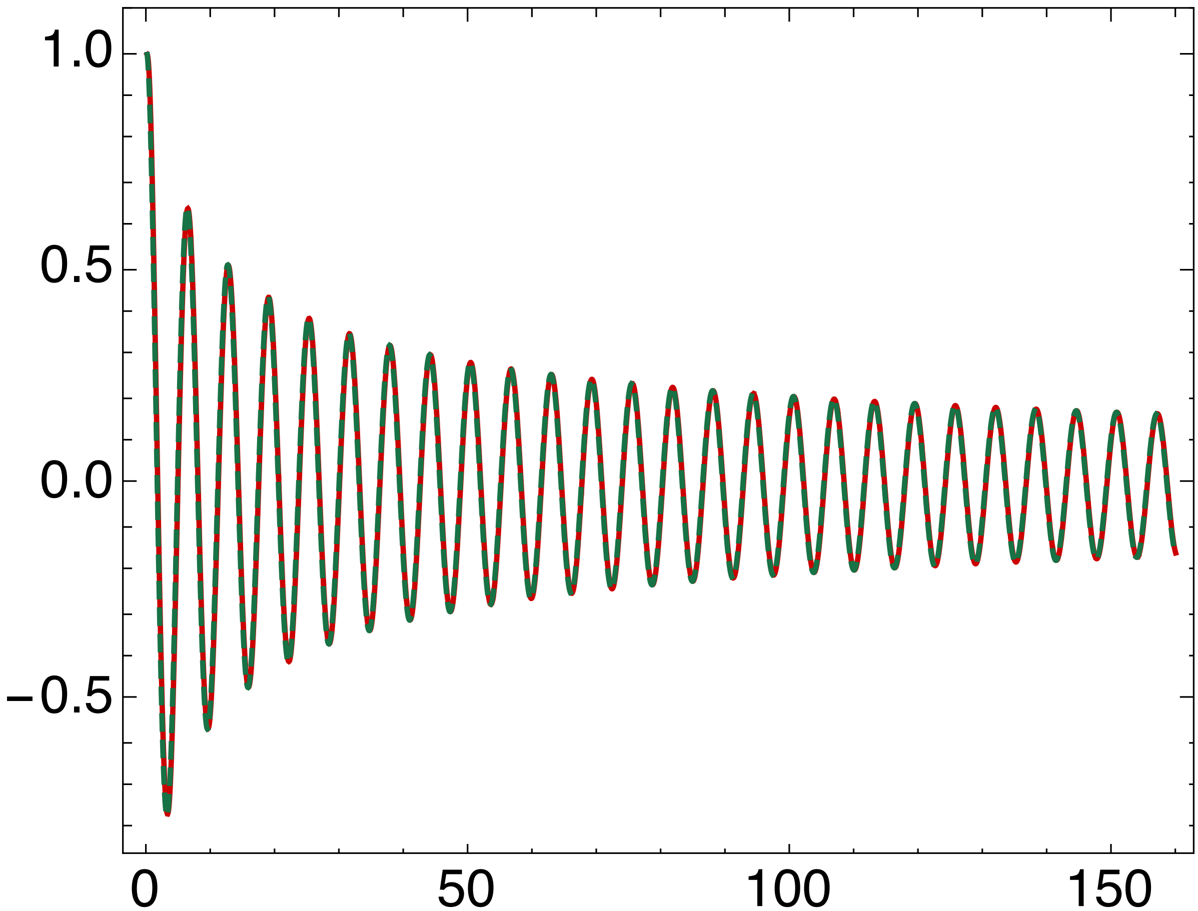

Figure 1:

Solution:

(a) The initial condition is has scale a_0 which defines a length scale. We set $m = \omega_0 = a_0 = 1$ this yields

$$\bar{x} = x/a_0 \tag{2}$$
$$\bar{t} = \omega_0 t \tag{3}$$
$$\bar{v} = v/(\omega_0 a_0) \tag{4}$$

The original equation is

$$m\ddot{x} + m\omega_0^2 x + \beta \dot{x}^3 = 0 \tag{5}$$

which after dividing by $m\omega_0^2 a_0$ becomes

$$\frac{d^2\overline{x}}{d\overline{t}^2} + \overline{x} + \epsilon \left(\frac{d\bar{x}}{d\bar{t}} \right)^3 = 0 \tag{6}$$

with

$$\epsilon = \frac{\beta(\omega_0 a_0)^3}{m\omega_0^2 a_0} \,. \tag{7}$$

This is the ratio of the viscous forces $\beta(\omega_0 a_0)^3$ to spring forces $m\omega_0^2 a_0$.

(b) We try a specific form for the zeroth order solution:

$$x^{(0)}(t) = A(t)\cos(-\omega_0 t + \varphi(t)) \tag{8}$$

where $\omega_0 = 1$ in practice. We keep it around for clarity. For simplicity we notate

$$\Omega(t) = -\omega_0 t + \varphi(t) \tag{9}$$

The full solution is

$$x(t) = x^{(0)}(t) + x^{(1)}(t) \tag{10}$$

Substituting into the equations we find

$$\ddot{x}^{(0)} + \omega_0^2 x^{(0)} = 2\dot{A}\omega_0 \sin(\Omega) + 2A\omega_0\dot{\varphi}\cos(\Omega) + O(\ddot{A}, \ddot{\varphi}) \tag{11}$$

Similarly for the nonlinear term

$$\epsilon v^3(t) = \epsilon(A\omega_0 sin(-\omega_0 t + \varphi))^3 + \mathcal{O}(\epsilon^2) \tag{12}$$
$$\simeq \frac{\epsilon A^3\omega_0^3}{-8i}\left(e^{i\Omega} - e^{-i\Omega}\right)^3 \tag{13}$$
$$= -\frac{\epsilon A^3\omega_0^3}{4}\left(\sin(3\Omega) - 3\sin(\Omega)\right) \tag{14}$$

So the equation of motion reads

$$\ddot{x}^{(1)} + \omega_0^2 x^{(1)} + \sin(\Omega)\left[2\dot{A}\omega_0 + \frac{3}{4}\epsilon A^3\omega_0^3\right] + \cos(\Omega)\left[2A\omega_0\dot{\varphi}\right] - \frac{A^2\omega_0^3}{4}\sin(3\Omega) = 0 \tag{15}$$

In order to ovoid secular term we choose (recalling that ω_0)

$$\frac{dA}{dt} = - \frac{3}{8}\epsilon A^3 \tag{16}$$

$$\frac{d\varphi}{dt} = 0 \tag{17}$$

Solving the first equation is easily solved with the boundary condition that $A = 1$ at $t = 0$:

$$\frac{dA}{A^3} = -\frac{3}{8}\epsilon dt\,. \tag{18}$$

Integrating we find

$$A = \frac{1}{(1 + \frac{3\epsilon}{4}t)^{1/2}} \tag{19}$$

The phase φ is constant, and this constant must be set to 0 in order that there is no initial velocity at $t = 0$. Thus our final solution is

$$x^{(0)} = \frac{1}{(1 + \frac{3\epsilon}{4}t)^{1/2}} \cos(t) \tag{20}$$

A comparison between the numerical solution (the solid red line) and the analytical form (Eq. (**??**)) is given in Fig. 1

Problem 2. (Likharev) An effective mass

For a system with the following Lagrangian: Consider an oscillator with generalized coordinate q and resonant frequency ω_0 with an effective mass which is weakly coordinate-dependent $m_{\text{eff}} = m(1 + \epsilon q^2)$ where m is a constant[2] . The lagrangian is

$$L = \frac{1}{2} m_{\text{eff}}(q)\dot{q}^2 - \frac{1}{2} m\omega_0^2 q^2 \tag{21}$$

Calculate the frequency of oscillations using secular perturbation theory, and from an integral given in class for the period of one dimensional systems (see "Motion of 1d systems" online). Assume that the amplitude of the oscillations is A and that $\epsilon A^2 \ll 1$.

You should find by both methods that to first order in ϵA^2

$$\omega \simeq \omega_0 \left(1 - \frac{A^2 \epsilon}{4}\right) . \tag{22}$$

Give a qualitative explanation for why the shift is negative.

[2] Technically m has units, $[m] = \text{kg} \cdot \text{m}^2$ and q is dimensionless

Solution:

The Lagrangian is

$$L = \frac{1}{2} m_{\rm eff}(q)\dot{q}^2 - \frac{1}{2} m\omega_0^2 q^2 \tag{23}$$

where $m_{\rm eff}(q) = m(1+\epsilon q^2)$. The equation of motion is

$$\partial_t(m(1+\epsilon q^2)\partial_t q) = -m\omega_0^2 q + m\epsilon q\dot{q}^2 \tag{24}$$

Here we write the zero-th order solution as

$$q^{(0)}(t) = A(t)\cos(\omega_0 t + \varphi(t))\,. \tag{25}$$

For a steady state solution (such as this) we are can consider a form with $A = \text{const}$ and $\varphi = \Delta\omega\, t$. Or simply

$$q^{(0)}(t) = A\cos(\omega t)\,, \tag{26}$$

with $\omega = \omega_0 + \Delta\omega$ to be adjusted so that no secular terms appear. The equation of motion without approximation can be writen:

$$\ddot{q} + \omega^2 q + \epsilon\left[\partial_t(q^2\partial_t q) - q\dot{q}^2\right] + (\omega_0^2 - \omega^2)q = 0 \tag{27}$$

Approximating $\omega_0^2 - \omega^2 \simeq -2\omega_0\Delta - \omega \simeq 2\omega\Delta\omega$ at first order in ϵ . The mass term is also approximated

$$\begin{aligned}\partial_t q^2\partial_t q - q(\partial_t q)^2 = \frac{A^3}{8}\partial_t\left[(e^{i\omega t}+e^{-i\omega t})^2\partial_t(e^{i\omega t}+e^{-i\omega t})\right] \\ - \frac{A^3}{8}\left[(e^{i\omega t}+e^{-i\omega t})(\partial_t(e^{i\omega t}+e^{-i\omega t}))^2\right]\end{aligned} \tag{28}$$

Or

$$\partial_t q^2\partial_t q - q(\partial_t q)^2 = -\frac{A^3\omega^2}{2}\cos(3\omega t) - \frac{A^3}{2}\omega^2\cos(\omega t) \tag{29}$$

The $\cos(3\omega t)$ term is not in resonance with the oscillator, and in the zeroth approximation can be neglected. It will of course be necessary in a first approxiamation to keep this term; it will fix $q^{(1)}$. Then we find that the equation of motion:

$$\frac{d^2q^{(1)}}{dt^2} + \omega^2 q^{(1)} - \underbrace{\left[2\omega_0\Delta\omega + \frac{\omega^2 A^2}{2}\epsilon\right]}_{\text{secular}-\text{term}} q^{(0)} - \frac{A^2}{2}\omega^2\epsilon\cos(3\omega t) = 0 \tag{30}$$

Leading to a frequency shift of

$$\Delta\omega = -\frac{A^2\epsilon}{4}\omega_0 \tag{31}$$

in order to cancel the secular term. In determining this shift we have ignored the difference between ω_0 and ω since the whole correction is already first order in ϵ.

Of course the frequency shift is negative, as the mass shift is $\propto \epsilon q^2$, resulting in an increase in mass on average. This lowers the frequency.

We can also use the first order integral to determine the period. Recall that for a Lagrangian

$$L = \frac{1}{2}m(q)\dot{q}^2 - U(q)\,, \tag{32}$$

the first integral is

$$E = \frac{1}{2}m(q)^2 + U(q)\,. \tag{33}$$

This may be inverted, determining the time evolution of the system:

$$t - t_0 = \int_{q_0}^{q} dq \left(\frac{m(q)}{2(E - U(q))}\right)^{1/2} \tag{34}$$

The turning points happens when

$$E = U(q) = \frac{1}{2}m\omega_0^2 q^2 \tag{35}$$

or

$$q_{\pm} = \pm\sqrt{\frac{2E}{m\omega_0^2}} \tag{36}$$

Then we may expand the mass

$$\sqrt{m(q)} = \sqrt{m}(1 + \epsilon q^2)^{1/2} \simeq \sqrt{m}(1 + \frac{1}{2}\epsilon q^2) \tag{37}$$

Leading to an approximate expression for the period

$$\tau = 2\int_{q_-}^{q_+} dq\,(\sqrt{m/2})\,\frac{1}{\sqrt{E - U(q)}} + 2\int_{q_-}^{q_+} dq\,(\sqrt{m/2})\frac{\epsilon q^2}{2}\frac{1}{\sqrt{E - U(q)}} \tag{38}$$

Defining a dimensionless integration variable

$$u = \frac{q}{\sqrt{2E/m\omega_0^2}} \equiv \frac{q}{A} \tag{39}$$

which is the amplitude in units of the maximum we find

$$\omega_0\tau = 2\int_{-1}^{1} du\,\frac{1}{\sqrt{1 - u^2}} + A^2\frac{\epsilon}{2}2\int_{-1}^{1} du\,\frac{u^2}{\sqrt{1 - u^2}} \tag{40}$$

The remaining integrals can be done using the β function (or gamma function) yielding 2π and $\pi/2$ respectively for a total shift

$$\tau = \frac{2\pi}{\omega_0}(1 + \frac{A^2\epsilon}{4}) \tag{41}$$

The (angular) frequency in agreement with before

$$\omega = \frac{2\pi}{\tau} \simeq \omega_0(1 - \frac{A^2\epsilon}{4} + \ldots) \tag{42}$$

Problem 3. Anharmonic oscillations to quadratic order

Consider the oscillator with energy E in the potential

$$U = \frac{1}{2} m\omega_0^2 q^2 + \frac{c}{3} q^3 \tag{43}$$

where the anharmonic contribution is small. The oscillator is at the top of its arc at $t = 0$. We will determine an approximation to $q(t)$

$$q(t) = q^{(0)} + q^{(1)} + q^{(2)} \tag{44}$$

to second order in c.

(a) Choose an appropriate set of units so that the equation of motion can be written with

$$\frac{d^2\bar{q}}{d\bar{t}^2} + \bar{q} + \bar{c}\,\bar{q}^2 = 0\,, \tag{45}$$

with initial condition $\bar{q}(0) = 1$. $\bar{q}$, $\bar{c}$ and $\bar{t}$ are dimensionless versions of q, c and t. To lighten the notation we will drop the bars for the remainder of this problem. $\bar{c}$ is small in this problem; what does this imply for c?

(b) Solve for $q^{(0)}$, $q^{(1)}$, and $q^{(2)}$. You should find to order c^2

$$q(t) = a\cos(\omega t) - \frac{a^2 c}{2} + \frac{a^2 c}{6}\cos(2\omega t) + \frac{a^3 c^2}{48}\cos(3\omega t) \tag{46}$$

with

$$\omega = 1 - \frac{5c^2}{12} + \ldots \tag{47}$$

and amplitude a adjusted to reproduce the initial condition $q(0) = 1$:

$$1 = a - \frac{a^2}{2}c + \frac{a^2 c}{6} + \frac{a^3 c^2}{48} \tag{48}$$

or

$$a(c) = 1 + \frac{1}{3}c + \frac{29}{144}c^2 + \ldots \tag{49}$$

Hint: A general approach is to try a zero-th order ansatz of the form

$$q^{(0)} = A(t)\cos(-\omega_0 t + \varphi(t)) \tag{50}$$

and this will always work. Here you could try a more restricted form, by making an anasatz for $A(t)$ and an ansatz for $\varphi(t)$, and write the zeroth order solution as[3]

$$q^{(0)} = A\cos(\omega t) \tag{51}$$

[3]In this ansatz we treat $A(t)$ as a constant, and write $\varphi(t) = -\Delta\omega t$. So the frequency is $\omega = \omega_0 + \Delta\omega$.

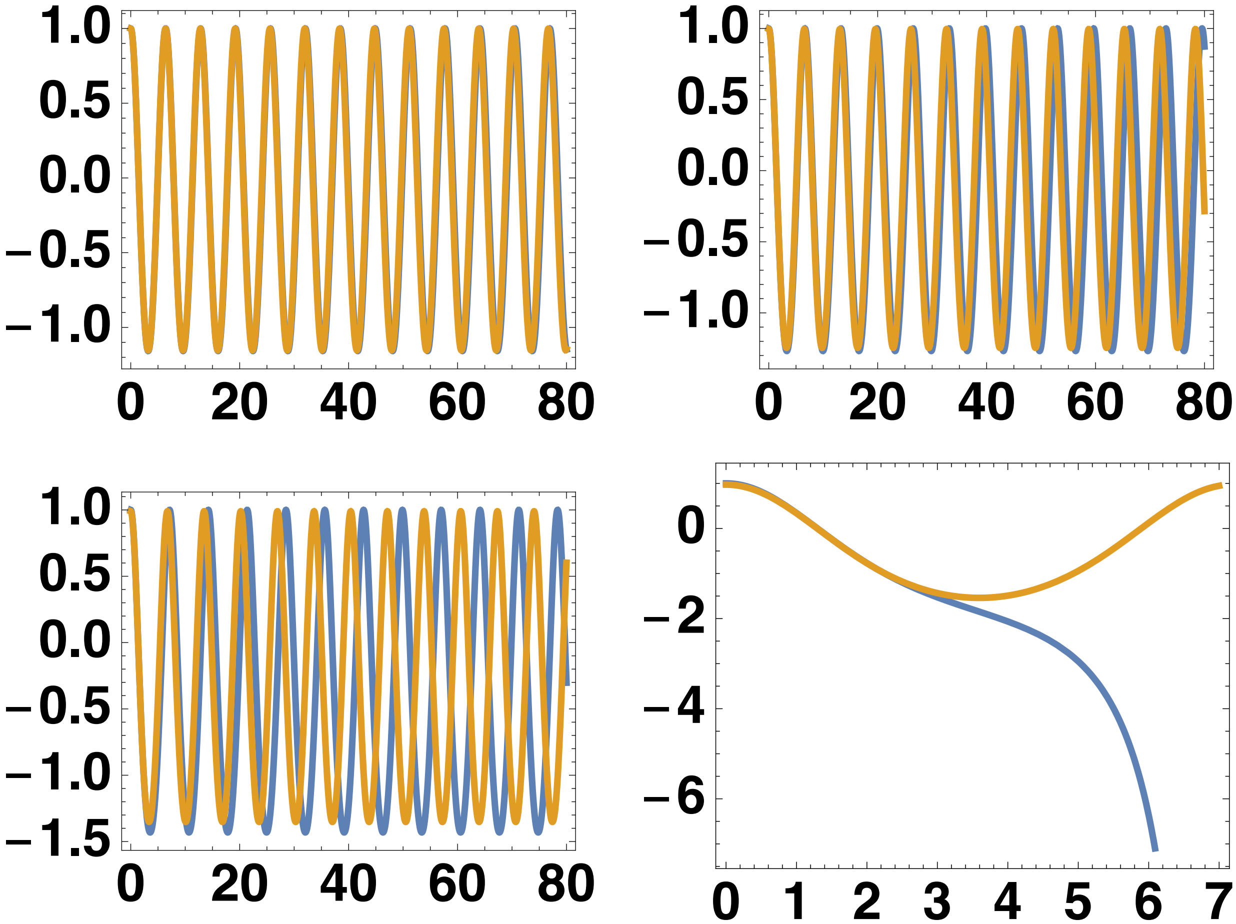

Figure 2: Analytical perturbative solution (yellow curve) compared to the numerical solution (blue curve) versus time for $q(t)$. Reading the graphs like words in a book, the comparison is for $c = 0.2, 0.3, 0.4$ and 0.55 (so $c = 0.3$ is the top right graph).

where A and ω are adjusted order by order to remove secular terms:

$$A = 1 + \lambda A_1 + \lambda^2 A_2 + \dots \tag{52}$$

$$\omega = 1 + \lambda \omega_1 + \lambda^2 \omega_2 + \dots \tag{53}$$

(Here λ is a book keeping parameter which denotes the order in $\bar{c}$, i.e. λ^2 denotes a term of order $\bar{c}^2$. We will set $\lambda = 1$ at the end.) One could hope that this ansatz will work here since the energy is fixed, and the period of the non-linear oscillations is also fixed. If the ansatz in Eq. (12) was not general enough, then there will be secular terms which can't be captured with this form, and then you would go back to the more general approach in Eq. (11).

(c) The graph in Fig 2 compares solution in Eq. (7) to a numerical solution. Explain why the perturbative solution fails qualitatively for $c = 0.55$.

(d) The motion is periodic with period T. Qualitatively sketch the power spectrum, i.e. if $q(t)$ is expanded in a Fourier series, $q(t) = \sum_n q_n e^{-i2\pi nt/T}$, sketch $|q_n|^2$ versus n. How does increasing the non-linearity c change this spectrum?

Solution

(a)

$$m\ddot{q} + m\omega_0^2 q + cq^2 = 0 \tag{54}$$

we can choose our units for mass, time, space independently: $m = 1, \omega_0 = 1, q_0 = 1$. This amounts to defining

$$\bar{q} = q/q_0 \qquad \bar{t} = \omega_0 t \tag{55}$$

Then the equation of motion factors

$$m\omega_0^2 q_0 \left[\ddot{\bar{q}} + \bar{q} + \bar{c}\bar{q}^2\right] = 0 \tag{56}$$

where $\bar{c} = cq_0^2/(m\omega_0^2 q_0)$, is the typical size of the ratio of the anharmonic forces to the harmonic forces. We of course must have $\bar{c} \ll 1$ for perturbation theory to be valid. We will dispense with the bars below.

(b) This is a classic rotation wave setup. We write

$$q = q^{(0)} + \lambda q^{(1)} + \lambda^2 q^{(2)} \tag{57}$$

where λ is a formal parameter which counts powers of c. We will set $\lambda = 1$ at the end. We write $q^{(0)} = A\cos(\omega t)$, where

$$\omega = 1 + \lambda\omega_1 + \lambda^2\omega_2 + \ldots \tag{58}$$

the frequency is also adjusted at each order. The amplitude is also adjusted

$$A = 1 + \lambda A_1 + \lambda^2 A_2 \tag{59}$$

to match our required initial condition, $q(0) = 1$.

Then the equation of motion is (without approximation) can be written

$$\ddot{q} + \omega^2 q + (1 - \omega^2)q + c\lambda q^2 = 0 \tag{60}$$

Here we have anticipated that the oscillation frequency will not be unity, and regrouped terms.

The frequency terms are approximated

$$(1 - \omega^2) \simeq -2\lambda\omega_1 + \lambda^2(-2\omega_2 + \omega_1^2) + \ldots \tag{61}$$

The non-linear term is approximated

$$c\lambda q^2 = \lambda c\lambda(q^{(0)})^2 + c\lambda^2(2q^{(0)}q^{(1)}) + \ldots \tag{62}$$

Of course we have

$$\ddot{q} + \omega^2 q = (\ddot{q}^{(0)} + \omega^2 q^{(0)}) + \lambda(\ddot{q}^{(1)} + \omega^2 q^{(1)}) + \lambda^2(\ddot{q}^{(2)} + \omega^2 q^{(2)}) + \ldots \tag{63}$$

Then at first order in c (or λ) we have

$$\lambda\ddot{q}^{(1)} + \omega^2\lambda q^{(1)} - 2\lambda\omega_1 q^{(0)} + c\lambda(q^{(0)})^2 = 0\,. \tag{64}$$

So since

$$(q^{(0)})^2 = A^2\cos^2(\omega t) = \frac{A^2}{2} + \frac{A^2}{2}\cos(2\omega t) \tag{65}$$

we have at first order

$$\ddot{q}^{(1)} + \omega^2 q^{(1)} + \underbrace{-2\omega_1 A\cos(\omega t)}_{\text{secular term}} + cA^2\left(\frac{1}{2} + \frac{1}{2}\cos(2\omega t)\right) \tag{66}$$

So we require that the secular term be zero yielding

$$\omega_1 = 0 \tag{67}$$

Then $q^{(1)}$ is easily solved

$$q^{(1)} = -\frac{cA^2}{2\omega^2} - \frac{cA^2\cos(2\omega t)}{-(2\omega)^2+\omega^2} = -\frac{cA^2}{2\omega^2} + \frac{cA^2}{6\omega^2}\cos(2\omega t) \tag{68}$$

The full solution takes the form at first order takes the form

$$q = A\cos(\omega t) - \frac{cA^2}{2\omega^2} + \frac{cA^2}{6\omega^2}\cos(2\omega t) \tag{69}$$

The second order second order

$$\lambda^2\ddot{q}^{(2)} + \lambda^2\omega^2 q^{(2)} + \lambda^2(-2\omega_2 + \omega_1^2)q + c\lambda^2(2q^{(0)}q^{(1)}) = 0\,. \tag{70}$$

Using the fact that $\omega_1 = 0$ we have

$$\ddot{q}^{(2)} + \omega^2 q^{(2)} - 2\omega_2 q^{(0)} + c(2q^{(0)}q^{(1)}) = 0\,. \tag{71}$$

Tackling the product

$$2cq^{(0)}q^{(1)} = -\frac{c^2A^3}{\omega^2}\cos(\omega t) + \frac{c^2A^3}{3\omega^2}\cos(\omega t)\cos(2\omega t) \tag{72}$$

Then last term is simplified using

$$\cos(\omega t)\cos(2\omega t) = \frac{1}{2}\cos(3\omega) + \frac{1}{2}\cos(\omega t) \tag{73}$$

for a total equation of motion of

$$\ddot{q}^{(2)} + \omega^2 q^{(2)} + \underbrace{(-2\omega_2 A + \frac{c^2A^3}{\omega^2} - \frac{c^2A^3}{6\omega^2})}_{\text{secular term}}\cos(\omega t) - \frac{c^2A^3}{6\omega^2}\cos(3\omega t) = 0 \tag{74}$$

Requiring that the secular term is absent at second order yields

$$\omega_2 = -\frac{5c^2A^2}{12\omega^2} \simeq -\frac{5c^2}{12} + \mathcal{O}(c^3) \tag{75}$$

where in the last step we have used that at zeroth order $A = 1$ and $\omega = 1$. The second order correction is

$$q^{(2)} = \frac{-c^2A^3}{6\omega^2}\frac{\cos(3\omega t)}{-(3\omega)^2 + \omega^2} = \frac{c^2A^3}{48\omega^2}\cos(3\omega t) \tag{76}$$

To summarize we find

$$q(t) = A\cos(\omega t) - \frac{cA^2}{2\omega^2} + \frac{cA}{6\omega^2}A\cos(2\omega t) + \frac{1}{48}\left(\frac{cA}{\omega^2}\right)^2 A\cos(3\omega t) \tag{77}$$

with

$$\omega = 1 - \frac{5c^2}{12} \tag{78}$$

as our approximate solution. The amplitude A should be adjusted to match the initial condition. In this case we have $q(0) = 1$, leading to

$$1 = A - \frac{A^2}{2}c + \frac{A^2c}{6} + \frac{A^3c^2}{48} + \mathcal{O}(c^3) \tag{79}$$

This may be solved iteratively for A

$$A(c) = 1 + \frac{1}{3}c + \frac{29}{144}c^2 + \ldots \tag{80}$$

It is also fine to exapand out the amplitudes in Eq. (**??**) in powers of c. It is however *not ok* to expand $\cos((1 - 5c^2/12)t)$ in powers of c since c^2t is of order unity at late times.

(c) Fig. 2, shows the analytical solution (yellow curve) together with the numerical curve (blue curve) for $c = 0.2, 0.3, 0.4, 0.55$. Clearly it works well for $c \simeq 0.3$. But it fails qualitatively for $c = 0.55$ (except at short times).

What is going on here is the following. If the energy is low enough, then the particle bounces around in the potential, and the approximate solution works well for a long period of time. However, when the energy gets sufficiently large, the particle can make it over the barrier and escape to infinity. This is shown in Fig. **??**. The approximate solution in this case can not be expected to work except for a very short period of time. This is what is seen in Fig. 2(d).

(d) The power spectrum is very characteristic. Each higher harmonic for weakly nonlinear oscillations carries less and less power.

$$|q_1|^2 = |q_{-1}|^2 \sim 1 \tag{81}$$

$$|q_2|^2 = |q_{-2}|^2 \sim c^2 \tag{82}$$

$$|q_3|^2 = |q_{-3}|^2 \sim (c^2)^2 \tag{83}$$

As the non-linearities increase, the higher harmonics become more important.

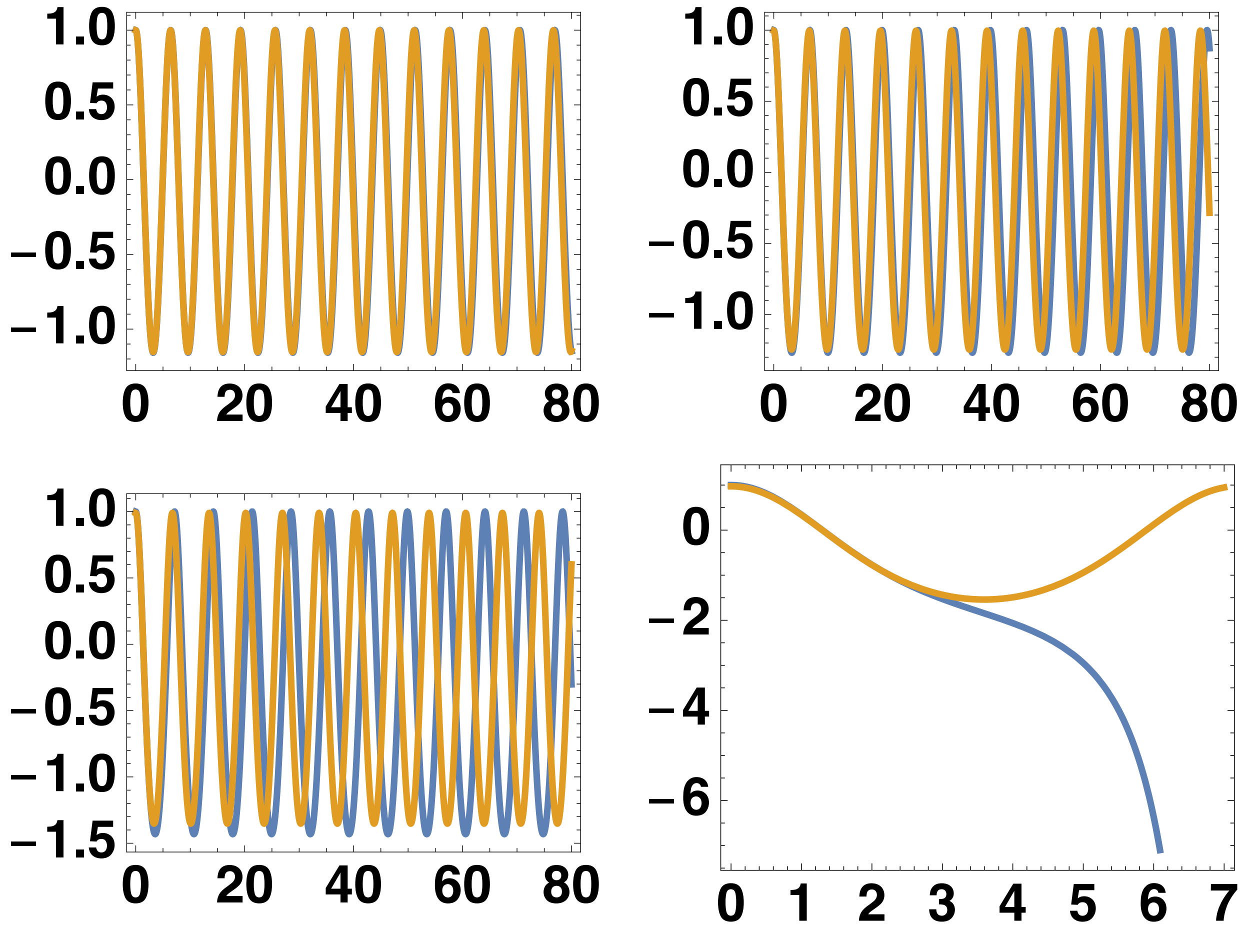

Figure 3: Analytical solution (yellow curve) compared to the numerical solution (blue curve) versus time. The comparison is for $c = 0.2, 0.3, 0.4, 0.55$ (read like a book).

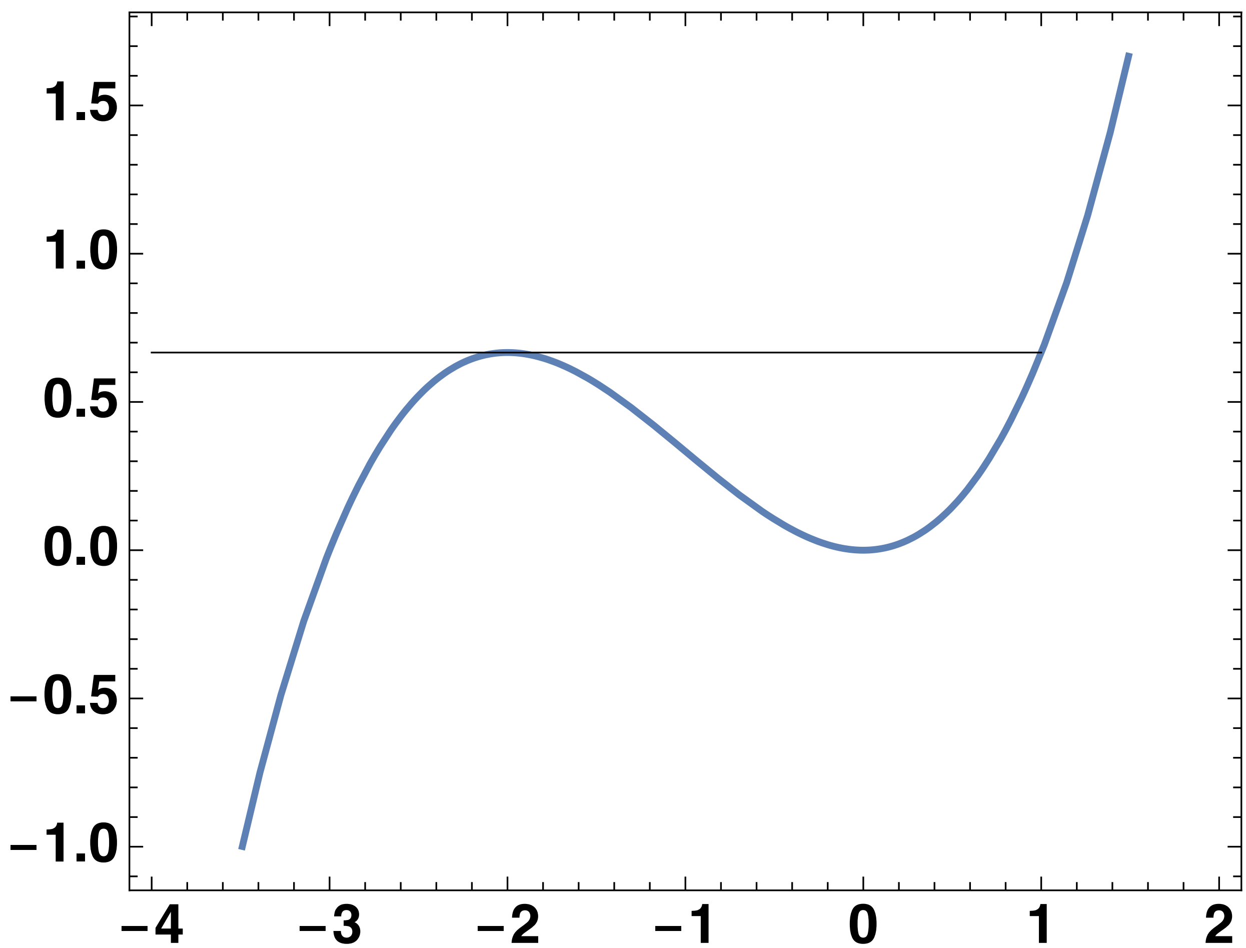

Figure 4: Potential (curve) and energy of particle (line) for $c = 0.5$