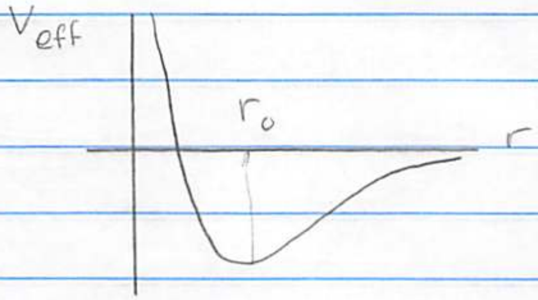
The Kepler Orbit: $U = -k/r$ (Goldstein 3.7)

- Before continuing let us analze the simple circular orbit:



$$V_{eff} = \frac{\ell^2}{2mr^2} + U(r)$$

$$\frac{\partial V_{eff}}{\partial r} = 0 \Rightarrow -\frac{\ell^2}{mr_o^3} + \frac{k}{r_o^2} = 0$$

Interpret using High School Physics!

Or $\boxed{r_o \equiv \frac{\ell^2}{mk} \equiv \text{the radius of the circular orbit with specified } \ell}$

- The potential energy for the circular orbit is

$$U_o = -\frac{k}{r_o} \equiv -2\mathcal{E}_o \quad \text{where} \quad \boxed{\mathcal{E}_o \equiv \frac{mk^2}{2\ell^2}}$$

- The kinetic energy for the circular orbit is:

$$T_o = \frac{\ell^2}{2\mu r_o^2} = \frac{mk^2}{2\ell^2} = +\mathcal{E}_o$$

KE is half as big and opposite sign as U_o.

This is the virial thrm

- So the energy is

$$\boxed{E = T + V = -\mathcal{E}_o \equiv \text{energy of circular orbit}}$$

- Now consider the integral for <u>Elliptic Orbits</u>

$$\phi - \phi_0 = \frac{\ell}{\sqrt{2\mu}} \int \frac{dr/r^2}{(E - V_{eff}(r))^{1/2}}$$

and switch to dimensionless variables. Measure r in units of r_0, and E and V_{eff} in units of $\mathcal{E}_0 = \frac{\ell^2}{2\mu r_0^2}$

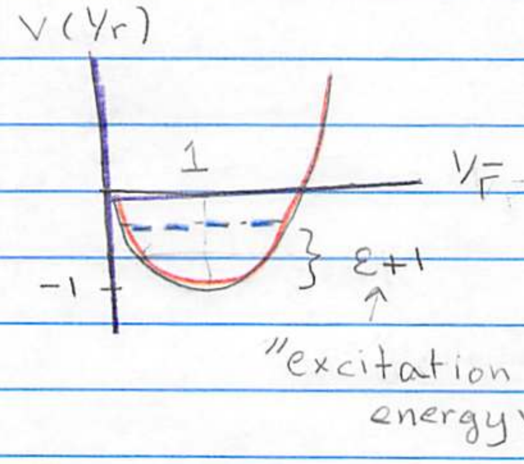
$$\bar{r} \equiv \frac{r}{r_0} \qquad \varepsilon \equiv \frac{E}{\mathcal{E}_0} \qquad V \equiv \frac{V_{eff}}{\mathcal{E}_0} = \frac{1}{\bar{r}^2} - \frac{2}{\bar{r}}$$

Watch!

$$= (1/\bar{r} - 1)^2 - 1$$

- So the integral becomes:

$$\phi - \phi_0 = \int \frac{dr/r^2}{\left[(\varepsilon + 1) - (1/\bar{r} - 1)^2\right]^{1/2}}$$



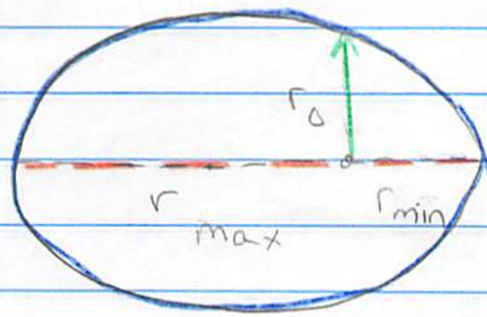
- Defining $u = 1/\bar{r}$ and integrating:

$$\phi - \phi_0 = \operatorname{acos}\left(\frac{u - 1}{\sqrt{\varepsilon + 1}}\right)$$

Or $u = 1 + \sqrt{1 + \varepsilon}\, \cos(\phi - \phi_0)$. Restoring units

$$\boxed{\frac{1}{r} = \frac{1}{r_0}\left(1 + \sqrt{1 + E/\mathcal{E}_0}\, \cos(\phi - \phi_0)\right)}$$

- This is the equation of an elipse with an origin as one of the focii:



$$\frac{1}{r} = \frac{1}{r_0}(1 + e\cos\phi)$$

- r_0 is known as the latus rectum

  e is the eccentricity

These two parameters determine all others

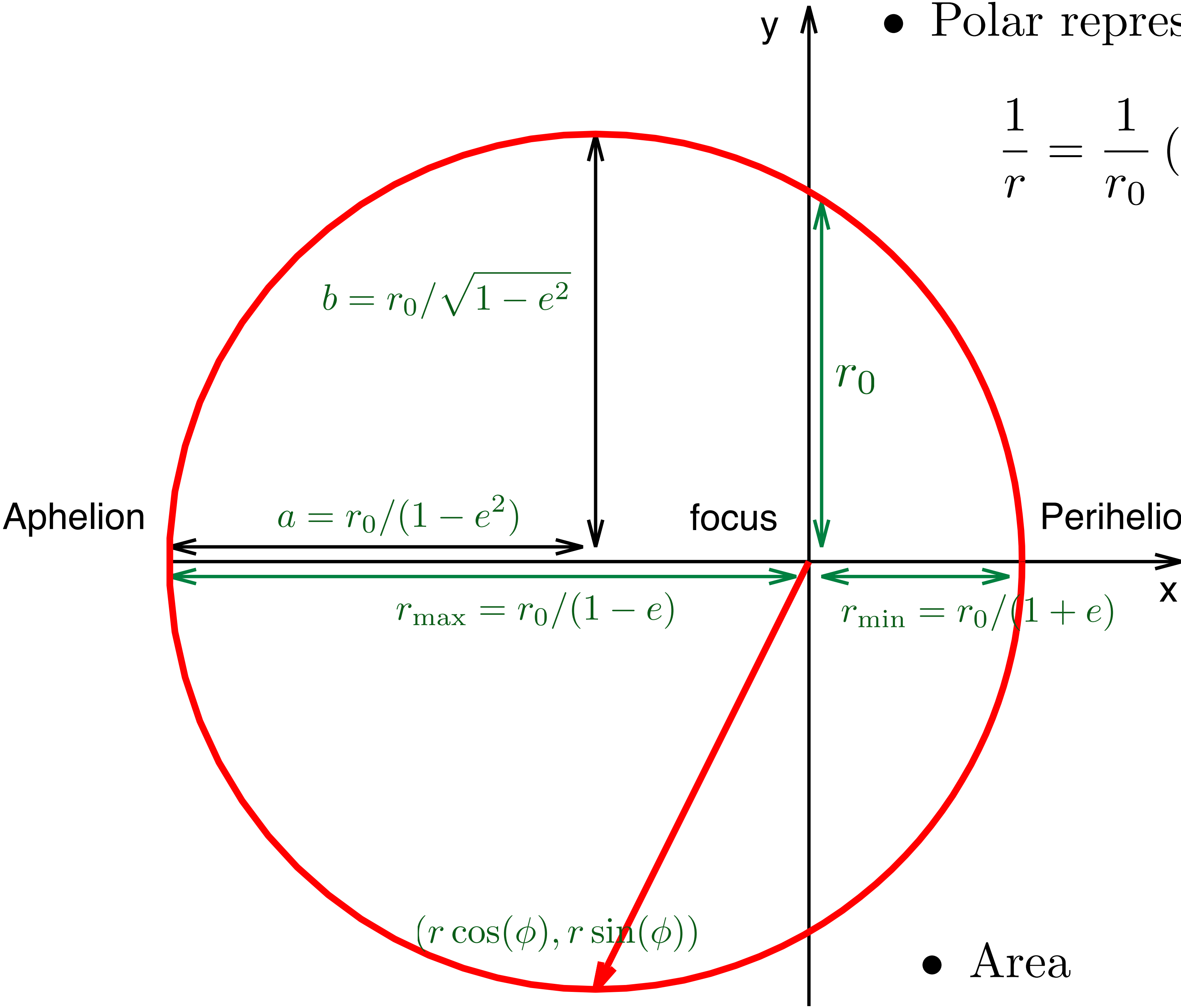
e.g $\frac{1}{r_{min}} = \frac{1}{r_0}(1+e)$ at $\phi = 0$

- These properties are related to the integrals of the motion:

$$r_0 = \frac{l^2}{\mu k} \qquad e = \sqrt{1 + E/\varepsilon_0} = (1 + 2El^2/\mu k)^{1/2}$$

- Polar representation

$$\frac{1}{r} = \frac{1}{r_0}\left(1 + e\cos(\phi)\right)$$



- Area

$$A = \pi a b$$

Videos are collected in different formats here: click me!

Keplerian Orbits: More Interesting Features

Click me!

(the perihelion is the short tip at rmin, see figure above)

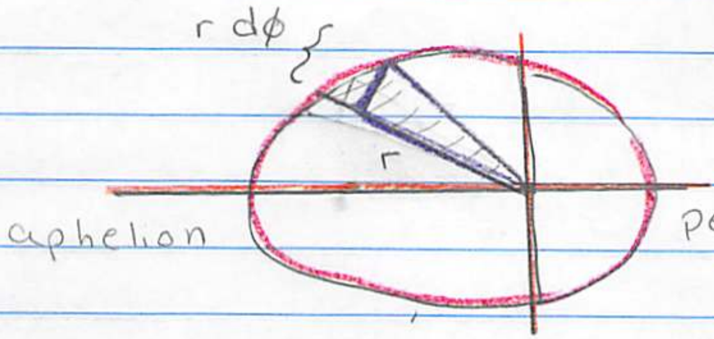
- Examining the movie we see that near the perihelion the planet is whipped around. This follows from Kepler's Law (a.k.a. angluar momentum conservation), which reads:

$$\ell = \mu r^2(\phi)\frac{d\phi}{dt} = \text{const}$$

const $= \ell/2\mu$

Or

$$\frac{1}{2}r^2(\phi)\, d\phi = (\text{const}) \times dt$$



$$\boxed{d(\text{Area}) = \frac{1}{2}r^2\, d\phi = \text{const} \times dt}$$

Kepler

- So, the particle sweeps out equal area per time, moving faster at the perihelion (where r is small) and slower at the apihelion (where r is large)

- The period is the area of the ellipse πab by $\frac{d(\text{Area})}{dt} = \frac{\ell}{2\mu} = \frac{1}{2}\sqrt{\frac{r_0 k}{\mu}}$:

Yielding

$$\boxed{T = 2\pi\, a^{3/2} \sqrt{\frac{\mu}{k}}}$$

Kepler's other law

$$\boxed{T^2 \propto a^3}$$

- A striking feature is the fact that the orbit is closed. Indeed for an orbit between r_{min} and r_{max} and back:

$$\Delta\phi = 2 \cdot \frac{\ell}{\sqrt{2\mu}} \int_{r_{min}}^{r_{max}} \frac{dr/r^2}{(E - V_{eff}(r))^{1/2}}$$

there and back

So unless the $\Delta\phi$ is a multiple of 2π the orbit will not be closed. Only by tuning the potential very carefully will this integral be exactly 2π.

Click me!

- Examine the movie of modified gravity. In this case the perihelion of the ellipse begins to precess. Note we have only modified gravity by a small amount

$$-\frac{k}{r} \longrightarrow \frac{k}{r_0}\left(\frac{r_0}{r}\right)^{1.1}$$

You will calculate the precession rate in homework

- Bertrand's Theorem

The only potentials of the form $U(r) \propto r^{\beta}$ which give closed orbits are for:

$\beta = 2$ (simple harmonic oscillator)

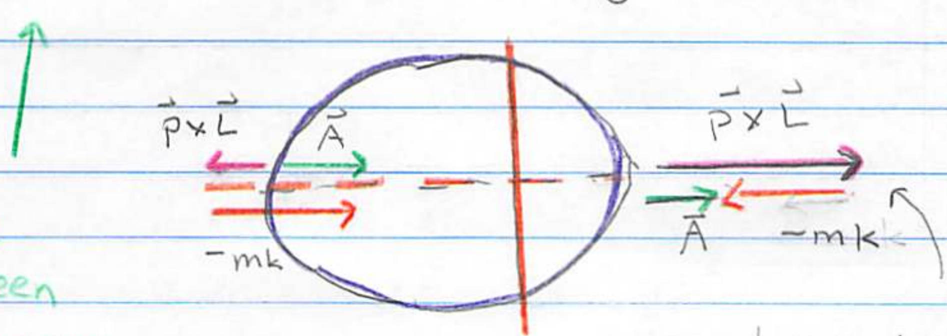
$\beta = -1$ (gravity)

- The reason why these keplerian orbits are closed is because of an additional conserved vector. The Laplace - Runge Lenz vector

$$\boxed{\vec{A} = \vec{p} \times \vec{L} - mk \frac{\vec{r}}{r}}$$

Runge - Lenz

It is difficult to come up with this, (maybe later), but it is not difficult to verify using the EOM that $d\vec{A}/dt = 0$. $\vec{A}$ is directed along the perihelion. It is constant in time so the perihelion can not precess.



the green vector

$\vec{p} \times \vec{L}$ is longer here because the particle is moving quickly here (Kepler)