General Coordinates - The advantage of Lagrangians

- Consider a general coordinates transformation

$$q_A = f_A(\vec{x}_a, t) \qquad a = 1, \dots, N \leftarrow \text{number of particles}$$

$\uparrow$ invertible $\qquad A = 1, \dots, 3N$ $\leftarrow$ Total # independent variable

e.g

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

- Choose a new lagrangian, $\underline{L}$, so that

$$L(\vec{x}, \dot{\vec{x}}, t) = \underline{L}(q, \dot{q}, t)$$

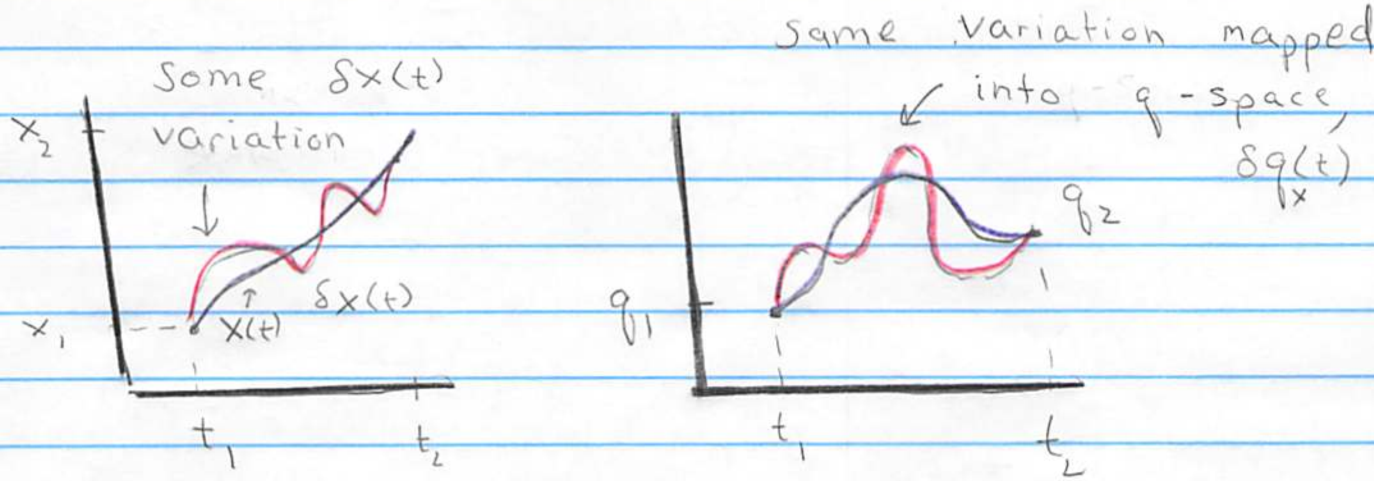
$\uparrow$ different functional form (so its techically a different func) but returns same value at corresponding points. We usually leave off the bar, but this can lead to confusion.

e.g

$$\underbrace{\frac{1}{2} m \dot{\vec{x}}^2}_{L(x, \dot{x}, t)} = \underbrace{\frac{1}{2} m (\dot{r}^2 + (r\dot{\theta})^2)}_{\underline{L}(q, \dot{q}, t)}$$

• So

$$\mathcal{S}[\vec{x}] = \int dt\, L(x, \dot{x}, t) = \int dt\, \underline{L}(q, \dot{q}, t) = \underline{S}[q]$$



So

$$S[\vec{x} + \delta\vec{x}] = \underline{S}[q + \delta q] = S[x] = \underline{S}[q]$$

i.e.

$$\boxed{S[q + \delta q] = \underline{S}[q]}$$

So the eom are the same and derived the same way. Briefly:

$$\underline{S}[q + \delta q] = \int dt\, \underline{L}\left(q + \delta q,\ \dot{q} + \frac{d\,\delta q}{dt},\ t\right)$$

And $\delta \underline{S} = \underline{S}[q + \delta q] - \underline{S}[q]$

$$\delta \underline{S} = \int dt\, \frac{\partial \underline{L}}{\partial q} + \frac{\partial L}{\partial \dot{q}} \frac{d}{dt}\delta q \Rightarrow \int dt \left(\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}}\right)\delta q$$

by parts

Leading to

$$\frac{\partial L}{\partial q} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = 0$$

Or in general for coordinates q_A $A = 1, \ldots 3N$

$$\boxed{\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^A} = \frac{\partial L}{\partial q^A}}$$

In general call

★ $p_A \equiv \frac{\partial L}{\partial \dot{q}^A}$ the momentum conjugate to the coordinate q^A

So

$$\frac{dp_A}{dt} = \frac{\partial L}{\partial q^A}$$

The derivative

$$\frac{\partial L}{\partial q^A}$$

is called the generalized force associated with coordinate q_A. If the Lagrangian does not depend on q_A we call this coordinate "cyclic". For cyclic coordinates the (generalized) force is zero, and the corresponding momentum p_A are constant in time. We love cyclic coordinates!